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Gorenstein Rings and Complete Interserections*

by

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1. Introduction

In this paper, we studied some properties of Gorenstein rings and complete intersections in the Cohen-Macaulay ring.

It is well known that they play an important part as Noetherian rings in the study of commutative algebra and algebraic geometry.

The study on Cohen-Macaulay rings has been remarkably developed by the concepts of depth, grade, regular sequence and Krull dimension([2],[6],[9],[18],[19]).

In particular, the study of relationship between injective dimension, global dimension of rings and modules and homological algebra has occupied most of the studies of Gorenstein rings([4],[16],[17]).

In addition, complete intersection as a special case of Gorenstein ring has been used in the study not of abelian varieties of dimension≥ 2 but of algebraic varieties in algebraic geometry([2],[12]).

Main theorems of this paper is Theorem 4.3. and 4.8..

The detailed contents are as follows:

In section 2, we describe the definition and some properties which are necessary in section 3,4. Lemma 2.1. and 2.2. are necessary in proving the main theorems.

In section 3, we prove some properties of Gorenstein rings and complete intersections. Theorem 3.7. proves that if a Noetherian ring A is a Gorenstein ring, then A[X] is a Gorenstein ring. In general, complete intersection is Gorenstein. Example 3.9. is a counter-example of the inverse of this theorem.

In section 4, which is the main part of this paper, we prove Theorem 4.3. and Theorem 4.8..

Theorem 4.8. is as follows:

Let (A, m) be a principal Noetherian local ring which is not a field. If Spec(A)

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is discrete, then $(A, \mathfrak{m})$ is a complete intersection.

2. Preliminaries

Let $(A, \mathfrak{m})$ be a Noetherian local ring. It follows that A is Artinian if and only if $\dim(A)=0$([1],[10],[13],[14],[15]). In this case, we have $\mathfrak{m}^n=0$ for some integer $n>0$.

Let Spec(A) be the prime spectrum of a commutative ring A with 1.

Lemma 2.1. Let A be a Noetherian ring. Then Spec(A) is discrete if and only if A is Artinian.

Proof. Suppose that Spec(A) is discrete. Then every element of Spec(A) is closed. That is, every prime ideal of A is maximal ideal. Hence, $\dim(A)=0$.

Conversely, if A is Artinian then Spec(A) is discrete since every prime ideal is maximal in an Artinian ring A. ///

Lemma 2.2. Let $(A, \mathfrak{m})$ be an Artinian local ring which is not a field. Then every ideal in A is principal if and only if $\dim_k(\mathfrak{m}/\mathfrak{m}^2)=1$, where k is the residue field $A/\mathfrak{m}$.

Proof. If every ideal in A is principal, then $\mathfrak{m}$ is principal ideal, and so $\dim_k(\mathfrak{m}/\mathfrak{m}^2)=1$.

Conversely, if $\dim_k(\mathfrak{m}/\mathfrak{m}^2)=1$, then $\mathfrak{m}$ is a principal ideal([1]), say $\mathfrak{m}=(x)$. Let $\mathfrak{a}$ be an ideal of A, other than (0) or (1). Then $\mathfrak{m}$ is nilradical, hence $\mathfrak{m}$ is nilpotent and therefore there exist an integer r such that $\mathfrak{a}\subseteq \mathfrak{m}^r$, $\mathfrak{a}\subsetneq \mathfrak{m}^{r+1}$; hence there exists $y\in\mathfrak{a}$ such that $y=ax^r$, $y\notin(x^{r+1})$; consequently, $a\notin(x)$ and a is a unit in A. Hence $x^r\in\mathfrak{a}$, $\mathfrak{m}^r=(x^r)\subseteq\mathfrak{a}$ and therefore $\mathfrak{a}=\mathfrak{m}^r=(x^r)$. Hence $\mathfrak{a}$ is principal. ///

Let A be a Noetherian semi-local ring and $\mathfrak{m}=\mathrm{rad}(A)$, the Jacobson radical of A. An ideal $\mathfrak{a}$ is called *an ideal of definition of* A if $\mathfrak{m}^\nu\subseteq\mathfrak{a}\subseteq\mathfrak{m}$ for some $\nu>0$. This is equivalent to saying that

$$\mathfrak{a}\subseteq\mathfrak{m}, \text{ and } A/\mathfrak{a} \text{ is Artinian}([11],[13],[14]).$$

Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension r. In this case, an ideal of definition of A and a primary ideal belonging to $\mathfrak{m}$ are same thing. Then we know that no ideal of definition are generated by less than r elements, and that there are ideals of definition generated by exactly r elements.

If $(x_1, \cdots, x_r)$ is an ideal of definition of A then we say that $\{x_1, \cdots, x_r\}$ is *a system of parameters of A*.

Definition 2.3. If there exists a system of parameters generating the maximal ideal $\mathfrak{m}$, then we say that A is *a regular local ring* and we call such a system of parameters *a regular system of parameters.*

In a Noetherian local ring $(A, \mathfrak{m})$, the number of elements of a minimal basis of $\mathfrak{m}$ is equal to $\mathrm{rank}_k \mathfrak{m}/\mathfrak{m}^2$ where $k=A/\mathfrak{m}$. We call $\mathrm{rank}_k \mathfrak{m}/\mathfrak{m}^2$ *the embedding dimension of A*, and denote it by em.dim(A). In general, we have

$$\dim(A) \leq \mathrm{em.dim}(A)$$

and the equality holds if and only if A is regular([11],[13],[14],[15]).

Definition 2.4. Let A be a commutative ring with 1 and let M be an A-module. Then $a_1, \cdots, a_n$ is said to be *an M-regular sequence* or simply *M-sequence* if it satisfies the following conditions:

(i) for each i $(1 \leq i \leq n)$, a_i is not a zero-divisor on

$$M/(a_1M + \cdots + a_{i-1}M), \text{ and}$$

(ii) $M \neq a_1M + \cdots + a_nM$.

If $\{a_1, \cdots, a_n\} \subseteq \mathfrak{a}$, an ideal of A, then we say $a_1, \cdots, a_n$ is *an M-regular sequence in* $\mathfrak{a}$. If there is no any $b \in \mathfrak{a}$ such that $a_1, \cdots, a_n, b$ is an M-regular sequence in $\mathfrak{a}$, then $a_1, \cdots, a_n$ is said to be *a maximal M-regular sequence in* $\mathfrak{a}$.

Let A be a commutative ring with 1, and let $x_1, \cdots, x_n$ be in A. Then the complex $K.$ is defined as follows:

(i) $K_0 = A$,

(ii) for each $p (1 \leq p \leq n)$, $K_p = \oplus A e_{i_1 \cdots i_p}$ is a free A-module of rank$\binom{n}{p}$ generated by the basis $\{e_{i_1 \cdots i_p} \mid 1 \leq i_1 < \cdots < i_p \leq n\}$.

(iii) for $p > n$, $K_p = 0$

(iv) the boundary (or differential) operator d: $K_p \longrightarrow K_{p-1}$ is defined by

$$d(e_{i_1 \cdots i_p}) = \sum_{r=1}^{p} (-1)^{r-1} x_{i_r} e_{i_1 \cdots \hat{i}_r \cdots i_p}$$

(if $p=1$, $d(e_i)=x_i$) where $\hat{i}_r$ indicates that i_r is omitted.

It is straightforward to check that $dd=0$. This complex is said to be *Koszul complex*, and it is denoted by

$$K.(x_1,\cdots,x_n),\ K.x,1\cdots n \text{ or } K.(\underline{x})$$

and for an A-module M we put

$$K.(\underline{x},M)=M\otimes_A K.(\underline{x}).$$

Let $x_1,\cdots,x_n$ be in A and let Ae_i be a free A-module of rank one with specified basis e_i for $i=1,\cdots,n$. Then

$$K.(x_i):\ 0\longrightarrow Ae_i\xrightarrow{x_i}A\longrightarrow 0$$

is the complex with

$$K_p(x_i)=\begin{cases}0 & (p\neq 0,1)\\ Ae_i\cong A & (p=1)\\ A & (p=0)\end{cases}$$

and $d(e_i)=x_i$. Hence we have

$$H_0(K.(x_i))=A/x_iA \text{ and } H_1(K.(x_i))=Ann(x_i).$$

Moreover, if we put

$$K.(x_1,\cdots,x_n)=K.(x_1)\otimes K.(x_2)\otimes\cdots\otimes K.(x_n)$$

then

$$Ae_{i_1\cdots i_p}=Ae_{i_1}\otimes\cdots\otimes Ae_{i_p}$$

and

$$e_{i_1\cdots i_p}=e_{i_1}\otimes\cdots\otimes e_{i_p}$$

for $1\leq i_1<\cdots<i_p\leq n$.

There is another interpretation of the Koszul complex.

Let $F=AX_1\oplus\cdots\oplus AX_n$ be a free A-module of rank n with a basis $\{X_1,\cdots,X_n\}$. Then the exterior product $\wedge^p F$ is a free module of rank $\binom{n}{p}$ with a basis $\{X_{i_1}\wedge\cdots\wedge X_{i_p}\,|\,1\leq i_1<\cdots<i_p\leq n\}$, so that there is an isomorphism of A-modules

$$\wedge^p F\longrightarrow K_p(\underline{x}),$$

which maps $X_{i_1}\wedge\cdots\wedge X_{i_p}$ to $e_{i_1\cdots i_p}$. Thus we can define

$$d:\ \wedge^p F \longrightarrow \wedge^{p-1}F$$

such that

$$d(X_{i_1}\wedge\cdots\wedge X_{i_p})=\sum_{r=1}^{p}(-1)^{r-1}X_{i_1}\wedge\cdots\wedge\hat{X}_{i_r}\wedge\cdots\wedge X_{i_p}.$$

If we adopt this definition, then we have to check $dd=0$([3], [9], [13]).

For any $x\in A$ and any complex $C.$, $C.\otimes_A K.(x)=C.(x)$ is the complex such that

$$(C.(x))_p=C_p\otimes_A A\oplus C_{p-1}\otimes_A Ae\cong C_p\oplus C_{p-1}$$

and thus we have an exact sequence of complexes:

$$0\longrightarrow C.\longrightarrow C.(x)\longrightarrow C.'\longrightarrow 0$$

where $C'_{p+1}=C_p$ for all $p\geq 0$.

For the boundary operator d of $C.$, the boundary operator d' of $C.(x)$ is defined by

$$d'(\xi,\eta)=(d\xi+(-1)^{p-1}\ x\eta,\ d\eta)$$

for each $(\xi,\eta)\in C_p\oplus C_{p-1}$. Therefore, the following hold.

$$\begin{aligned} d'(\xi,\eta)=0 &\Longrightarrow d\eta=0 \text{ and } d\xi=(-1)^p x\eta \\ &\Longrightarrow x(\xi,\eta)=(x\xi,\ x\eta) \\ &\qquad\qquad =d'(0,(-1)^p\xi)\in d'((C.(x))_{p+1}) \end{aligned}$$

Thus

$$xH_p(C.(x))=0 \text{ for all } p=0,1,2,\cdots\cdots.$$

Moreover, for each $\eta\in C'_p=C_{p-1}$ with $d\eta=0$ in $(C.(x))_p=C_p\oplus C_{p-1}$ we have

$$d'(0,\eta)=((-1)^{p-1}x\eta,\ 0).$$

Hence, from the exact sequence of complexes and by the above statements we have a long exact homology sequence,

$$\begin{aligned}(*);\ \cdots\longrightarrow H_p(C.)\longrightarrow H_p(C.(x))\longrightarrow H_{p-1}(C.)\xrightarrow{(-1)^{p-1}x}H_{p-1}(C.)\\ \longrightarrow H_{p-1}(C.(x))\longrightarrow\cdots\cdots\end{aligned}$$

and

$$xH_p(C.(x))=0 \text{ for all } p=0,1,2,\cdots\cdots.$$

Proposition 2.5. Let A be a commutative ring with 1, M an A-module and $x_1,\cdots,x_n$ be a sequence of elements of A.

(i) If $(\underset{\sim}{x})=(x_1,\cdots,x_n)$, then

$$(\underset{\sim}{x})H_p(\underset{\sim}{x},M)=0 \text{ for all } p=0,\ 1,\ 2,\cdots\cdots.$$

(ii) If $x_1,\cdots,x_n$ is an M-sequence, then

$$H_p(\underset{\sim}{x},M)=0 \text{ for } p\geq 1,\ H_0(\underset{\sim}{x},M)=M/\sum_{i=1}^{n}x_iM$$

where $H_p(\underset{\sim}{x},M)=H_p(K.(\underset{\sim}{x},M))$.

Proof. (i) Suppose $(\underset{\sim}{x}')=(x_1,\cdots,x_{n-1})$ and $K.(x_1,\cdots,x_{n-1})\otimes_A M=K.(\underset{\sim}{x}',M)$. Then

$$K.(\underset{\sim}{x}',M)\otimes_A K.(x_n)\cong K.(\underset{\sim}{x},M).$$

In this case, by(*)

$$x_n\cdot H_p(K.(\underset{\sim}{x}',M))\otimes_A K.(x_n)\cong x_n\cdot H_p(\underset{\sim}{x},M)=0$$

for all $p=0,1,2,\cdots\cdots$.

Similarly, for $1\leq i\leq n$ we have

$$x_i\cdot H_p(\underset{\sim}{x},M)=0 \text{ for all } p=0,1,2,\cdots\cdots.$$

Therefore, we have

$$(\underset{\sim}{x})H_p(\underset{\sim}{x},M)=0 \text{ for all } p=0,1,2,\cdots\cdots.$$

(ii) We shall prove the result by induction on n.

First,

$$H_1(x,M)=\{\xi\in M\,|\,x\xi=0\}=0$$

since x is an M-regular element, and thus the result is true when $n=1$.

Next, we assume that (ii) is true for all $1,2,\cdots,n-1$. Then by(*), we have the exact sequence

$$0=H_p(x_1,\cdots,x_{n-1},M)\longrightarrow H_p(x_1,\cdots,x_n,M)\longrightarrow H_{p-1}(x_1,\cdots,x_{n-1},M)=0$$

for $p>1$. Therefore

$$H_p(\underline{x}, M)=0 \text{ for } p>1.$$

We put $M_i=M/(x_1, \cdots, x_i)M$. Then

$$H_1(x_1, \cdots, x_{n-1}, M)=0 \longrightarrow H_1(\underline{x}, M) \longrightarrow H_0(x_1, \cdots, x_{n-1}, M)=M_{n-1} \xrightarrow{\pm x_n} M_{n-1} \longrightarrow \cdots$$

is exact when $p=1$. Since x_n is an M_{n-1}-regular element, $H_1(\underline{x}, M)$ must be zero.///

Let $(A, \mathfrak{m})$ be a Noetherian local ring with the minimal basis $\{x_1, \cdots, x_n\}$ of $\mathfrak{m}$. Put $E_*=K.(x_1, \cdots, x_n)$. Then, by (i) of Proposition 2.5.,

$$\mathfrak{m}\cdot H_p(E_*)=0 \text{ for all } p=0, 1, 2, \cdots\cdots.$$

Thus, each $H_p(E_*)$ is a k-vector space where $k=A/\mathfrak{m}$. If we put

$$\varepsilon_p(A)=\dim_k H_p(E_*)$$

then $\{\varepsilon_p | p=0, 1, 2, \cdots\}$ is a set of invariants of A. If A is regular, then it is clear that $\varepsilon_p(A)=0$ for $p\geq 1$ and $\varepsilon_0(A)=1$.

Let $(R, \mathfrak{n})$ be a regular local ring, and let $A=R/\mathfrak{a}$ for an ideal $\mathfrak{a}$ of R. Let

$$\underline{\xi}=\{\xi_1, \cdots, \xi_n\}$$

be a minimal basis of $\mathfrak{n}$. The Koszul complex for R and $\underline{\xi}$ is denoted by

$$K.(\underline{\xi}):\ 0 \longrightarrow L_n \longrightarrow \cdots\cdots \longrightarrow L_1 \longrightarrow L_0 \longrightarrow 0$$

Put $k=R/\mathfrak{n}$. Then

$$0 \longrightarrow L_n \longrightarrow \cdots\cdots \longrightarrow L_0 \longrightarrow k \longrightarrow 0$$

is a projective resolution over k as an R-module. Next

$$E_*=K.(\underline{\xi})\otimes_R A: 0 \longrightarrow L_n\otimes_R A \longrightarrow \cdots \longrightarrow L_1\otimes_R A \longrightarrow L_0\otimes_R A \longrightarrow 0$$

is a semi-complex. Therefore we have

$$H_p(E_*)=H_p(K.(\underline{\xi})\otimes_R A)=\mathrm{Tor}_p^R(k, A)$$

for all $p\geq 0$.

Proposition 2.6. In the above situation, if $\mu(\mathfrak{a})$ is the least number of generators of $\mathfrak{a}$, then $\mu(\mathfrak{a})=\dim_k H_1(E_*)=\varepsilon_1(A)$.

Proof. Consider the following exact sequence:

$$0 \longrightarrow \mathfrak{a} \longrightarrow R \longrightarrow A \longrightarrow 0.$$

Then we have the long exact sequence of homology groups

$$0 = \mathrm{Tor}_1{}^R(k, R) \longrightarrow \mathrm{Tor}_1{}^R(k, A) \longrightarrow k \otimes_R \mathfrak{a} \longrightarrow k \otimes_R R \xrightarrow{\cong} k \otimes_R A \longrightarrow 0$$

Since $k \otimes_R R \cong k \cong k \otimes_R A$,

$$\mathrm{Tor}_1{}^R(k, A) \cong k \otimes_R \mathfrak{a} \cong \mathfrak{a}/\mathfrak{n}\mathfrak{a} \text{ and } \mathrm{Tor}_1{}^R(k, A) = H_1(E_*)$$

Hence

$$\mu(\mathfrak{a}) = \dim_k(H_1(E_*)) = \varepsilon_1(A). \ ///$$

Let A be a commutative ring with 1 and $\mathfrak{a}$ an ideal of A. Then we have natural homomorphisms

$$\cdots \longrightarrow A/\mathfrak{a}^3 \longrightarrow A/\mathfrak{a}^2 \longrightarrow A/\mathfrak{a},$$

which make $\{A/\mathfrak{a}^n | n>0\}$ into an inverse system of rings. The inverse limit (or projective limit) ring $\varprojlim A/\mathfrak{a}^n$ is denoted by $\hat{A}$ and is called *the completion of A with respect to* $\mathfrak{a}$ *or the* $\mathfrak{a}$*-adic completion of A with* $\mathfrak{a}$*-adic topology* (*or linear topology*). So by the universal property we obtain a homomorphism $A \longrightarrow \hat{A}$. Similarly, if M is any A-module, we define $\hat{M} = \varprojlim M/\mathfrak{a}^n M$, and called it *the* $\mathfrak{a}$*-adic completion of M*. If $A = \hat{A}$, then A is said to be *complete*. Thus we know the following properties.

Proposition 2.7. Let A be a commutative ring with 1, $\mathfrak{a}$ an ideal of A and let $\hat{A}$ be the $\mathfrak{a}$-adic completion of A. Then

(i) $\hat{\mathfrak{a}} = \varprojlim \mathfrak{a}/\mathfrak{a}^n$ is an ideal of A. For any n,
$(\hat{\mathfrak{a}})^n = \mathfrak{a}^n \hat{A}$, and $\hat{A}/(\hat{\mathfrak{a}})^n = (A/\mathfrak{a}^n)\hat{}$;

(ii) If M is an A-module, $\hat{M}$ the $\mathfrak{a}$-adic completion of M, then $\hat{M} = M \otimes_A \hat{A}$;

(iii) If A is a Noetherian and M is a finitely generated A-module, then $\hat{M}$ is A-flat.

Proof. See ([1], [15]).

Proposition 2.8. Let $(A, \mathfrak{m})$ be a Noetherian local ring and $\hat{A}$ its completion. Then

(i) $\varepsilon_p(A)=\varepsilon_p(\hat{A})$ for all $p\geq 0$;

(ii) $\varepsilon_1(A)\geq$ em. dim(A)-dim(A) if A is homomorphic image of a regular local ring.

Proof. (i) Since the minimal basis of $\mathfrak{m}$ is also the minimal basis of $\mathfrak{m}\hat{A}$, if the Koszul complex for A is E_*, then we have the Koszul complex $E_*\otimes_A\hat{A}$ for $\hat{A}$. Thus

$$H_p(E_*)\otimes_A\hat{A}=H_p(E_*\otimes_A\hat{A})$$

since $\hat{A}$ is A-flat, and from $\mathfrak{m}\cdot H_p(E_*)=0$, $\varepsilon_p(A)=\varepsilon_p(\hat{A})$.

(ii) By (i), we may assume that A is complete. Since A is homomorphic image of a regular local ring, say $(R,\mathfrak{n})$, such that $A=R/\mathfrak{a}$.

$$\dim(R)=\text{em.}\dim(A) \quad ([2],[8],[14]).$$

Then

$$\varepsilon_1(A)=\mu(\mathfrak{a})\geq \text{ht}(\mathfrak{a})=\dim(R)-\dim(A)$$
$$=\text{em.}\dim(A)-\dim(A). \quad ///$$

Definition 2.9. Let $(A,\mathfrak{m})$ be a Noetherian local ring. If

$$\varepsilon_1(A)=\text{em.}\dim(A)-\dim(A)$$

then A is called *a complete intersection* (briefly C.I.).

Let A be a commutative ring with 1 and let $x_1,\cdots,x_n$ be an M-regular sequence in an ideal $\mathfrak{a}$ of A. If $x_1,\cdots,x_n$ is a maximal M-regular sequence in $\mathfrak{a}$, then the length of a maximal M-regular sequence in $\mathfrak{a}$ is said to be the *$\mathfrak{a}$-depth of M* and it is denoted by $\text{depth}_{\mathfrak{a}}(M)$ or by depth$(\mathfrak{a},M)$.

When $(A,\mathfrak{m})$ is a local ring, $\text{depth}_{\mathfrak{m}}(M)$ is written as depth(M) or $\text{depth}_A(M)$ and call it simply the *depth of M*.

Definition 2.10. Let $(A,\mathfrak{m})$ be a Noetherian local ring and M a finitely generated A-module. We say that M is *Cohen-Macaulay*(briefly C.M.) if $M=0$ or if depth(M) $=\dim(M)$ (in general depth$(M)\leq\dim(M)$). If the local ring A is C.M. as an A-module, then we call A a *Cohen-Macaulay ring*.

Let $(A,\mathfrak{m})$ be a C.M. local ring. Then the following are equivalent for every sequence $a_1,\cdots,a_r$ in $\mathfrak{m}$([9],[13]):

(i) the sequence $a_1,\cdots,a_r$ is A-regular,

(ii) ht $(a_1,\cdots,a_i)=i$ $(1\leq i\leq r)$,

(iii) ht $(a_1,\cdots,a_r)=r$,

(iv) there exist $a_{r+1}, \cdots, a_n$ ($n=\dim(A)$) in $\mathfrak{m}$ such that $\{a_1, \cdots, a_n\}$ is a system of parameters of A.

3. Gorenstein Rings and Complete Intersections

Let A be a commutative ring with 1 and let $\mathfrak{a}$ be an ideal of A. We say that $\mathfrak{a}$ is irreducible, if $\mathfrak{a}=I\cap J$ for some ideals I and J of A then either $\mathfrak{a}=I$ or $\mathfrak{a}=J$. Otherwise, $\mathfrak{a}$ is said to be reducible.

Proposition 3.1. Let $(A, \mathfrak{m})$ be a Noetherian local ring with $\dim(A)=n$ and $k=A/\mathfrak{m}$. Then the following conditions are equivalent:

(i) inj. $\dim(A)<\infty$;

(i)′ inj. $\dim(A)=n$;

(ii) $\mathrm{Ext}_A{}^i(k, A)$ $\begin{cases} =0 & (i\neq n) \\ \cong k & (i=n); \end{cases}$

(iii) there exists $i>n$ such that $\mathrm{Ext}_A{}^i(k, A)=0$;

(iv) $\mathrm{Ext}_A{}^i(k, A)$ $\begin{cases} =0 & (i<n) \\ \cong k & (i=n); \end{cases}$

(iv)′ A is a $C.M.$ ring and $\mathrm{Ext}_A{}^n(k, A)\cong k$;

(v) A is a $C.M.$ ring and every ideal of system of parameters of A is irreducible;

(v)′ A is a C.M. ring and there exists an ideal of system of parameters of A.

Proof. The proof can be found in ([14]).

Definition 3.2. If $(A, \mathfrak{m})$ is a Noetherian local ring satisfying one of the conditions in the above proposition, then $(A, \mathfrak{m})$ is called *Gorenstein*. If A is a Noetherian ring, and $A_{\mathfrak{p}}$ is Gorenstein for all $\mathfrak{p}\in\mathrm{Spec}(A)$ ($\Longleftrightarrow A_{\mathfrak{m}}$ is Gorenstein for all maximal ideal $\mathfrak{m}$ of A), then A is said to be a *Gorenstein ring*.

Let A be a Noetherian semi-local ring and let $\mathfrak{a}$ be an ideal of definition of A. For a finitely generated A-module M, $d(M)$ is called *the degree of Hilbert polynomial of M with respect to* $\mathfrak{a}$. We have the following properties ([1], [13]):

Proposition 3.3. Let A be a Noetherian semi-local ring, $\mathfrak{m}=\mathrm{rad}(A)$ and $M(\neq 0)$ a finitely generated A-module. Then

$$d(M)=\dim(M)$$

, and it is the smallest integer r such that there exist elements $x_1, \cdots, x_r$ of $\mathfrak{m}$ satisfying $l(M/x_1M+\cdots+x_rM)<\infty$.

Proposition 3.4. Let A, M and $\mathfrak{m}$ be as before. If $x\in\mathfrak{m}$, then

$$d(M)\geq d(M/xM)\geq d(M)-1.$$

Let A be a commutative ring with 1 and M an A-module. We put

$$\mathrm{Supp}(M)=\{\mathfrak{p}\in\mathrm{Spec}(A)\mid M_{\mathfrak{p}}\neq 0\}$$

, the support of M, and

$$V(\mathfrak{a})=\{\mathfrak{p}\in\mathrm{Spec}(A)\mid \mathfrak{a}\subseteq\mathfrak{p}\}$$

, and we say that a prime ideal $\mathfrak{p}$ of A is *an associated prime of* M if one of the following equivalent conditions holds:

(i) there exists an element $x\in M$ with $\mathrm{Ann}(x)=\mathfrak{p}$;

(ii) M contains a submodule isomorphic to $A/\mathfrak{p}$.

The set of the associated primes of M is denoted by $\mathrm{Ass}_A(M)$ or by $\mathrm{Ass}(M)$.

Lemma 3.5. Let $(A, \mathfrak{m})$ be a Noetherian local ring and let M be a finitely generated A-module.

(i) If $x_1, \cdots, x_r$ is an M-regular sequence in $\mathfrak{m}$ and $M'=M/\underline{x}M$, then

$$M \text{ is } C.M. \Longleftrightarrow M' \text{ is } C.M.$$

(ii) If M is $C.M.$, then for every $\mathfrak{p}\in\mathrm{Spec}(A)$ the $A_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ is C.M., and if $M_{\mathfrak{p}}\neq 0$ we have

$$\mathrm{depth}_{\mathfrak{p}}(M)=\mathrm{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}).$$

Proof. (i) By Nakayama's lemma we have $M=0$ if and only if $M'=0$. Suppose $M\neq 0$. Then it suffices to prove that $\dim(M')=\dim(M)-r$. By Proposition 3.3. and 3.4., we have

$$\dim(M')\geq\dim(M)-r.$$

On the other hand, suppose f is an M-regular element. We have

$$\begin{aligned}\mathrm{Supp}(M/fM)&=\mathrm{Supp}(M\otimes_A A/fA)=\mathrm{Supp}(M)\cap\mathrm{Supp}(A/fA)\\&=\mathrm{Supp}(M)\cap V(f)\end{aligned}$$

, and f is not in any minimal element of Supp(M), in other words, $V(f)$ does not contain any irreducible component of Supp(M). Hence

$$\dim(M/fM)<\dim(M).$$

This proves $\dim(M')\leq\dim(M)-r$.

(ii) We may assume that $M_{\mathfrak{p}}\neq 0$. Hence $\mathfrak{p}\supseteq \mathrm{Ann}(M)$. We know that

$$\dim(M_{\mathfrak{p}})\geq \mathrm{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})\geq \mathrm{depth}_{\mathfrak{p}}(M).$$

So we will prove $\mathrm{depth}_{\mathfrak{p}}(M)=\dim(M_{\mathfrak{p}})$ by induction on $\mathrm{depth}_{\mathfrak{p}}(M)$.

If $\mathrm{depth}_{\mathfrak{p}}(M)=0$, then $\mathfrak{p}$ is contained in some $\mathfrak{p}'\in \mathrm{Ass}_A(M)$, but $\mathrm{Ann}(M)\subseteq\mathfrak{p}\subseteq\mathfrak{p}'$ and the associated primes of M are the minimal prime over ideal of Ann(M). Hence $\mathfrak{p}=\mathfrak{p}'$, and $\dim(M_{\mathfrak{p}})=0$.

Next, suppose $\mathrm{depth}_{\mathfrak{p}}(M)>0$: take an M-regular element $x\in\mathfrak{p}$ and put $M_1=M/xM$. Since the localization preserves the exactness, the element x is $M_{\mathfrak{p}}$-regular. Therefore we have

$$\dim(M_1)_{\mathfrak{p}}=\dim(M_{\mathfrak{p}}/xM_{\mathfrak{p}})=\dim(M_{\mathfrak{p}})-1$$

and

$$\mathrm{depth}_{\mathfrak{p}}(M_1)=\mathrm{depth}_{\mathfrak{p}}(M)-1.$$

Since M_1 is $C.M.$ by (i), by induction hypothesis we have

$$\dim(M_1)_{\mathfrak{p}}=\mathrm{depth}_{\mathfrak{p}}(M_1).$$

This complete the proof of (ii). ///

Theorem 3.6. Let $(A,\mathfrak{m})$ be a Noetherian local ring, and let $x_1,\cdots,x_r$ be an A-regular sequence, and put $B=A/x_1A+\cdots+x_rA$. Then

A is Gorenstein if and only if B is Gorenstein.

Proof. By Lemma 3.5. (i), A is $C.M.$ if and only if B is $C.M.$. Suppose that A is Gorenstein. Then if the A-regular sequence $x_1,\cdots,x_r$ is extended to a maximal A-regular sequence $x_1,\cdots,x_r,x_{r+1},\cdots,x_n$ in $\mathfrak{m}$ where $\dim(A)=n$, $\{x_1,\cdots,x_n\}$ is a system of parameters of A, that is, $(x_1,\cdots,x_n)=\mathfrak{q}$ is an ideal of definition of A. Hence $\mathfrak{q}$ is $\mathfrak{m}$-primary and $\mathfrak{q}$ is irreducible since A is Gorenstein.

Let ϕ: $A\longrightarrow B=A/x_1A+\cdots+x_rA$ be the natural projection. Then $(\bar{x}_{r+1},\cdots,\bar{x}_n)=\overline{\mathfrak{q}}$

is $\bar{\mathfrak{m}}$-primary and $\{\bar{x}_{r+1}, \cdots, \bar{x}_n\}$ is a system of parameters of B. Hence it suffices to prove that $\bar{\mathfrak{q}}$ is irreducible. If $\bar{\mathfrak{q}}=\bar{I}\cap\bar{J}$ for some ideals $\bar{I}$ and $\bar{J}$ of B, then there exist ideals I and J such that

$$I=\bar{I}\cap A,\quad J=\bar{J}\cap A$$

and

$$\mathfrak{q}=\bar{\mathfrak{q}}\cap A=(\bar{I}\cap\bar{J})\cap A=I\cap J.$$

But since $\mathfrak{q}$ is irreducible, it follows that $\mathfrak{q}=I$ or $\mathfrak{q}=J$. Hence $\bar{\mathfrak{q}}=\bar{I}$ or $\bar{\mathfrak{q}}=\bar{J}$. Therefore, $\bar{\mathfrak{q}}$ is irreducible.

Conversely, similarly we can prove that if B is Gorenstein then A is Gorenstein. ///

Theorem 3.7. If a Noetherian ring A is a Gorenstein ring, then a polynomial ring $A[X]$ is a Gorenstein ring.

Proof. Let P be a prime ideal of $A[X]$ and let $\mathfrak{p}=P\cap A$. Then $A[X]_P=(A_\mathfrak{p}[X])_P$. Hence we may assume that $(A,\mathfrak{m})=(A_\mathfrak{p},\mathfrak{p}A_\mathfrak{p})$ is Gorenstein and $\dim(A)=0$ by Theorem 3.6., We put $P\cap A=\mathfrak{m}$ and $A[X]_P=B$. Then it suffices to prove that B is Gorenstein.

Let PB be the unique maximal ideal of a local ring B such that $PB\cap A[X]=P$ and $PB\cap A=\mathfrak{m}$. Hence $\mathfrak{m}\cdot A[X]\subseteq P$.

Therefore we have two cases:

Case I. $P=\mathfrak{m}\cdot A[X]$.

We know that $\mathfrak{m}$ is nilpotent, that is, $\mathfrak{m}^r=0$ for some $r>1$ since $(A,\mathfrak{m})$ is a local ring and $\dim(A)=0$. Hence $\dim(A[X]_{\mathfrak{m}A(X)})=0$ since $\mathrm{ht}(P)=0$.

Case II. $P=\mathfrak{m}A[X]+f(X)A[X]$ for $f(X)\notin\mathfrak{m}\cdot A[X]$, $f(X)\in A[X]$.

Thus $A[X]/\mathfrak{m}A[X]\cong k[X]$ is a principal ideal domain where $k=A/\mathfrak{m}$.

Then $\overline{f(X)}\in k[X]$ is irreducible.

Hence we have $\dim(B)=\dim(A[X]_P)=1$, since $\mathrm{ht}(P)=1$, $f(X)$ is B-regular since $f(X)$ is a non-zero divisor of B. Thus $f(X)$ is a system of parameter element of B.

We put $C=B/(f)$, then C is a free A-module with a finite rank. In fact, if $f(X)=X^n+a_1X^{n-1}+\cdots+a_n$, then C is generated by $\{1, X, \cdots, X^{n-1}\}$ over A. Hence $C=A\oplus AX\oplus\cdots\oplus AX^{n-1}\cong A^n$ and $\mathfrak{m}C$ is a maximal ideal of C. Therefore

$$\begin{aligned}\mathrm{Hom}_C(C/\mathfrak{m}C, C)&\cong\mathrm{Hom}_A(k, A)\otimes_A C\\&\cong A/\mathfrak{m}\otimes_A C\\&\cong C/\mathfrak{m}C\end{aligned}$$

Thus if $\mathrm{Ext}_A{}^i(k, A)=0$ for some i, then $\mathrm{Ext}_C{}^i(C/mC, C)=0$ and if $\mathrm{Ext}_A{}^i(k, A)\cong k$ for some i, then $\mathrm{Ext}_C{}^i(C/mC, C)\cong C/mC$. Hence if A is Gorenstein then C is Gorenstein, and thus if C is Gorenstein then B is Gorenstein by Theorem 3.6. ///

Proposition 3.8. Let A be a Noetherian local ring. Then the following hold:

(i) A is a complete intersection iff $\hat{A}$ is a complete intersection.

(ii) Let A be a complete intersection and let R be a regular local ring such that $A=R/\mathfrak{a}$ for some ideal $\mathfrak{a}$ of R. Then $\mathfrak{a}$ is generated by an R-regular sequence.

Conversely, if $\mathfrak{a}$ is such an ideal of R then $R/\mathfrak{a}$ is a complete intersection,

(iii) If A is complete intersection, then A is Gorenstein.

Proof. The proof can be found in [14].

Hence we know the following hierarchy of Noetherian local ring;

$$C.I. \Longrightarrow \text{Gorenstein} \Longrightarrow C.M.$$

Example 3.9. $A=k[[X, Y, Z]]/(X^2-Y^2, Y^2-Z^2, XY, YZ, XZ)$ is Gorenstein, but A is not $C.I.$, where k is a field.

Proof. Let $I=(X^2-Y^2, Y^2-Z^2, XY, YZ, XZ)$ be an ideal of $k[[X,Y,Z]]$. We put $A=k[x,y,z]$ where x, y and z are the image of X, Y and Z under the canonical homomorphism π: $k[[X,Y,Z]]\longrightarrow A$. Then $x=\pm y$, $y=\pm z$ and $xy=yz=xz=0$ in A.

If $M=(X,Y,Z)$ is a maximal ideal of $k[[X, Y, Z]]$, then $M^3\subseteq I$ since $X^3=(X^2-Y^2)X+(XY)Y$ is in I. Similarly Y^3, Z^3 are in I. Hence I is an M-primary and $\mathrm{ht}(I)=3$. Thus $\dim(A)=0$ and A is generated by $\{1, x, y, z, x^2\}$ over k.

The formal power series ring $k[[X, Y, Z]]$ over k is $C.M.$ ([13]) and I is M-primary, that is, I is an ideal of definition of $k[[X, Y, Z]]$, generated by a system of parameters of a regular sequence X^2-Y^2, Y^2-Z^2, XY, YZ, XZ. Hence $k[[X, Y, Z]]$ is Gorenstein, and so A is Gorenstein by Theorem 3.6.. But $\mathrm{ht}(I)=3$ and

$$\mu(I)=\dim_k(I/MI)\leq 5 \cdots\cdots\cdots (1)$$

On the other hand, if ϕ: $I/MI\longrightarrow M^2/M^3$, then the basis of M^2/M^3 is $\{X^2, Y^2, Z^2, XY, YZ, XZ\}$ and the basis of $Im\ \phi$ is $\{X^2-Y^2, Y^2-Z^2, XY, YZ, XZ\}$. Hence

$$\mu(I)=\dim_k(I/MI)\geq 5 \cdots\cdots\cdots (2)$$

Therefore, we have $\mu(I)=5$ by (1) and (2). Thus $\mathrm{ht}(I)\neq\mu(I)$. Hence A is not $C.I.$.///

4. Main Theorems

Let A be a commutative ring and let M be an A-module. If a sequence of A-modules

$$A^p \longrightarrow A^q \longrightarrow M \longrightarrow 0$$

is exact, then M is called *of finite presentation*. Hence if M is of finite presentation, we have an exact sequence

$$0 \longrightarrow K \longrightarrow N \longrightarrow M \longrightarrow 0$$

where K is finitely generated if N is finitely generated.

Lemma 4.1. Let A be a commutative ring with 1 and let M be a finitely generated A-module. If M is of finite presentation, then

$$U_F = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid M_\mathfrak{p} \text{ is free } A_\mathfrak{p}\text{-module}\}$$

is open in $\mathrm{Spec}(A)$.

Proof. If $\{w_1, \cdots, w_r\}$ is a basis of $M_\mathfrak{p}$ for $\mathfrak{p} \in \mathrm{Spec}(A)$, then $w_i = m_i/s_i$ for some $m_i \in M$ and $s_i \in S = A/\mathfrak{p}$, and $A^r \longrightarrow M$ is surjective. Thus there exists an open set $D(f)$ at $\mathfrak{p}$ such that $D(f) = \bigcap_{i=1}^{n} D(s_i)$ for some $f \in A$ since $\{D(a) \mid a \in A\}$ is an open basis of $\mathrm{Spec}(A)$.

We may assume that $A = A_f$ and $M = M \otimes_A A_f \cong M_f$. Then for every $\mathfrak{q} \in D(f)$ $\{w_1, \cdots, w_r\}$ generates $M_\mathfrak{q}$. Hence for every $\mathfrak{q} \in \mathrm{Spec}(A) = \mathrm{Spec}(A_f)$, $M_\mathfrak{q} = \sum A_\mathfrak{q} w_i$ since $D(f) = \mathrm{Spec}(A)$.

If ϕ: $A^r \longrightarrow M$ is defined by $\phi(a_1, \cdots, a_r) = \sum a_i w_i$ then ϕ is surjective. Hence if $\ker \phi = K$, then we have

$$0 \longrightarrow K \longrightarrow A^r \longrightarrow M \longrightarrow 0$$

is exact, and $K_\mathfrak{q} = 0$ for all $\mathfrak{q} \in D(f)$ and $(A_\mathfrak{q})^r = M_\mathfrak{q}$. Hence $D(f) \subseteq U_F$. Therefore U_F is open in $\mathrm{Spec}(A)$. ///

Lemma 4.2. Let A be a Noetherian local ring and let $\mathfrak{a}$ be a proper ideal of A. If $\mathrm{proj.dim}(\mathfrak{a}) < \infty$, then the following holds.

$\mathfrak{a}$ is generated by A-sequence $\Longleftrightarrow \mathfrak{a}/\mathfrak{a}^2$ is free $A/\mathfrak{a}$-module([14], [18]).

Theorem 4.3. Let R be a regular ring and $\mathfrak{a}$ be an ideal of R. If $A=R/\mathfrak{a}$, then

$$\{\mathfrak{p}\in\operatorname{Spec}(A)\,|\,A_{\mathfrak{p}} \text{ is } C.\,I.\}$$

is open in Spec(A).

Proof. Let $\mathfrak{p}$ be a prime ideal of A. Then $P/\mathfrak{a}=\mathfrak{p}$ for some prime ideal P of R. By Proposition 3.8.(ii), $A_{\mathfrak{p}}$ is complete intersection if and only if $\mathfrak{a}_P$ is generated by an R_P-regular sequence.

Since $A_{\mathfrak{p}}=R_P/\mathfrak{a}_P=(R/\mathfrak{a})_{\mathfrak{p}}$ is a local ring, R_P is a regular local ring and R_P is Gorenstein if and only if $A_{\mathfrak{p}}$ is Gorenstein, and so proj. dim$(\mathfrak{a})<\infty$. Hence by Lemma 4.2., $\mathfrak{a}_P/\mathfrak{a}_P^2$ is a free $A_{\mathfrak{p}}$-module. Therefore by Lemma 4.1.,

$$\{\mathfrak{p}\in\operatorname{Spec}(A)\,|\,A_{\mathfrak{p}} \text{ is } C.\,I.\}$$

is open in Spec(A). ///

Proposition 4.4. A regular local ring $(A,\mathfrak{m})$ is a Unique Factorization Domain (UFD) ([11],[13],[14],[15]).

Proposition 4.5. A Noetherian domain A is a UFD iff every prime ideal of A with height 1 is principal.

Proof. Suppose that A is a UFD and that $\mathfrak{p}$ is a prime ideal of A with ht$(\mathfrak{p})=1$. Then for any non-zero element $a\in\mathfrak{p}$, if $a=\prod_{i=1}^{n}\pi_i$ where π_i is prime element, then there exists π_i such that $\pi_i\in\mathfrak{p}$ since $\mathfrak{p}$ is a prime ideal. Hence $(\pi_i)\subseteq\mathfrak{p}$, and (π_i) is a non-zero prime ideal with height 1. Therefore $\mathfrak{p}=(\pi_i)$.

Conversely, assume that every prime ideal of A with height 1 is principal. Since A is Noetherian, every element of A which neither 0 nor unit is a finite product of irreducible elements of A. Let x be an irreducible element of A. If $\mathfrak{p}$ is a minimal prime over ideal of (x), then ht$(\mathfrak{p})=1$. Hence $\mathfrak{p}=(y)$ for some $y\in A$ from the hypothesis, and $(x)\subseteq\mathfrak{p}=(y)$. Thus if $x=yc$ for some $c\in A$, then c is unit since x is irreducible. Hence $(x)=(y)=\mathfrak{p}$. ///

Proposition 4.6. If $(A,\mathfrak{m})$ is a complete local ring, then A is a homomorphic image of a regular local ring ([14]).

Lemma 4.7. Let $(A,\mathfrak{m})$ be a Noetherian local ring. If A is a Cohen-Macaulay ring and em. dim$(A)=\dim(A)+1$, then A is a complete intersection,

Proof. Let $\hat{A}$ be the completion of A with respect to $\mathfrak{m}$.

By Proposition 3.8.(i), A is a complete intersection iff $\hat{A}$ is a complete intersection. Hence we may assume that A is a complete Noetherian local ring. Thus by Proposition 4.6., there exists a regular local ring R such that $A = R/\mathfrak{a}$ where $\mathfrak{a}$ is an ideal of R. Hence

$$\dim(R) = \text{em.}\dim(A) = \dim(A) + 1.$$

Since $\dim(R) = \dim(A) + \text{ht}(\mathfrak{a})$, $\text{ht}(\mathfrak{a}) = 1$.
Since R is a UFD by Proposition 4.4. and $\text{ht}(\mathfrak{a}) = 1$, $\mathfrak{a}$ is a principal ideal by Proposition 4.5.. Thus A is a complete intersection. ///

Theorem 4.8. Let $(A, \mathfrak{m})$ be a principal Noetherian local ring which is not a field (that is, $\mathfrak{m} \neq (0)$). If Spec(A) is discrete, then $(A, \mathfrak{m})$ is a complete intersection.

Proof. By Lemma 2.1., $(A, \mathfrak{m})$ is a Artinian local ring. Hence $(A, \mathfrak{m})$ is a Cohen-Macaulay ring since $0 \leq \text{depth}(A) \leq \dim(A) = 0$.
Therefore $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1$ by Lemma 2.2., and we have

$$1 = \text{em.}\dim(A) = \dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim(A) + 1.$$

Hence $(A, \mathfrak{m})$ is a complete intersection by Lemma 4.7.. ///

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