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BANACH ALGEBRAS OF YEH–FEYNMAN AND FRESNEL INTEGRABLE FUNCTIONALS *

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1. INTRODUCTION

In his paper [4], J.S. Chang introduced the analytic Yeh–Feynman integral, defined by analytic continuation of the Yeh–Wiener integral, and found a Banach algebra $\mathcal{S}(L_2(Q))$ of functionals on Yeh–Wiener space which are a type of stochastic Fourier transform of complex Borel measures on $L_2(Q)$, where $Q = [0, a] \times [0, b]$ is a fixed rectangle in two dimensional Euclidean space $\mathbf{R}^2$, and $L_2(Q)$ will denote the space of real–valued Lebesgue measurable, square integrable functions on Q. And he showed [4; Theorem 4.1] that the analytic Yeh–Feynman integral exists for all elements of $\mathcal{S}(L_2(Q))$, and obtained a formula for this Yeh–Feynman integral.

In a monograph [1], Albeverio and Høegh–Krohn introduced the Fresnel integral on a real separable infinite–dimensional Hilbert space, and studied the Banach algebra $\mathcal{F}(H)$ of Fresnel integrable functionals. The main purpose of this paper is to find a Banach algebra of Fresnel integrable functionals which is isometrically isomorphic to the Banach algebra $\mathcal{S}(L_2(Q))$, and then establish the relationship between the analytic Yeh–Feynman integral and the Fresnel integral. Here the Fresnel integral is a slight extension of that defined by Albeverio and Høegh–Krohn. Finally we obtain a translation theorem for our Fresnel integral, and then, using this theorem and Theorem 3.3, we obtain a translation theorem for the analytic Yeh–Feynman integral which is identical with Yoo's result [18; Theorem 5.2].

2. PRELIMINARIES

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Let $Q = [0,a] \times [0,b]$ be a fixed rectangle in the two dimensional Euclidean space $\mathbf{R}^2$ and let $C(Q)$ be the space of real-valued continuous functions on Q. $C_2(Q)$ will denote the Yeh–Wiener space, that is, the space of functions x in $C(Q)$ such that $x(s,0) = x(0,t) = 0$ for $0 \le s \le a$ and $0 \le t \le b$. m will denote the Yeh–Wiener measure on $C_2(Q)$.

A subset E of Yeh–Wiener space is said to be *scale-invariant measurable* if ρE is Yeh–Wiener measurable for every $\rho > 0$. A scale-invariant set N is said to be *scale-invariant null* if $m(\rho N) = 0$ for every $\rho > 0$. A property which holds except on a scale-invariant null set is said to hold *scale-invariant almost everywhere* (s-a.e.). It is well known that the class of scale-invariant measurable sets forms a σ-algebra. A functional F on $C_2(Q)$ is said to be *scale-invariant measurable* if it is measurable with respect to this σ-algebra. For a complete discussion of scale-invariant measurability in Yeh–Wiener space see [5].

DEFINITION 2.1. *Let F be a complex-valued functional on $C_2(Q)$ which is s-a.e. defined and scale-invariant measurable and which is such that the Yeh–Wiener integral*

$$J(\lambda) = \int_{C_2(Q)} F(\lambda^{-\frac{1}{2}} x)dm(x)$$

exists for all $\lambda > 0$. If there exists a function $J^*(\lambda)$ analytic in $\mathbf{C}^+ = \{\lambda \in \mathbf{C} : Re(\lambda) > 0\}$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the analytic Yeh–Wiener integral of F over $C_2(Q)$ with parameter λ, and, for λ in $\mathbf{C}^+$, we write

$$\int_{C_2(Q)}^{\mathrm{any}\,\omega_\lambda} F(x)dm(x) = J^*(\lambda).$$

Let q be a non-zero real parameter and let F be a functional whose analytic Yeh–Wiener integral exists for λ in $\mathbf{C}^+$. If the following limit exists, we call it the analytic Yeh–Feynman integral of F over $C_2(Q)$ with parameter q, and we write

$$\int_{C_2(Q)}^{\mathrm{any}\,f_q} F(x)dm(x) = \lim_{\lambda \to iq} \int_{C_2(Q)}^{\mathrm{any}\,\omega_\lambda} F(x)dm(x)$$

where λ approaches $-iq$ through $\mathbf{C}^+$.

We shall say that two functionals $F(x)$ and $G(x)$ are *equal s–a.e.*, denoted by $F \approx G$, if for each $\rho > 0$ the equation $F(\rho x) = G(\rho x)$ holds for m–a.e. x in $C_2(Q)$. Equality s–a.e. is an equivalence relation for functionals on $C_2(Q)$. It is the appropriate relation for the analytic Yeh–Feynman integral [4].

Now we define the Banach algebra $\mathcal{S}(L_2(Q))$. Let $\mathcal{M}(L_2)$ be the collection of complex–valued, countably additive, Borel measures on $L_2(Q)$ with finite total variation. With convolution taken as multiplication, $\mathcal{M}(L_2)$ becomes a commutative Banach algebra.

DEFINITION 2.2. *Let σ be in $\mathcal{M}(L_2)$. Consider the functional $\tilde{\sigma}$ defined for s–a.e. x in $C_2(Q)$ by the formula*

$$\tilde{\sigma}(x) = \int_{L_2(Q)} \exp\{i \int_Q v(s,t)\tilde{d}x(s,t)\} d\sigma(v), \tag{2.1}$$

where $\int_Q v(s,t)\tilde{d}x(s,t)$ is the Paley–Wiener–Zygmund (P.W.Z.) integral which is a simple type of stochastic integral. For a complete discussion of P.W.Z. integral see [14].

An element of $\mathcal{S}(L_2(Q))$ is an equivalence class $[\tilde{\sigma}]$ of functionals which are s–a.e. equal to $\tilde{\sigma}$ for some σ in $M(L_2)$.

REMARK 2.1. *J.S. Chang showed [4, Theorem 3.1] that the correspondence $\sigma \longrightarrow [\tilde{\sigma}]$ is injective. And converting convolution to pointwise multiplication, and letting $\|[\tilde{\sigma}]\| = \|\sigma\|$, he showed [4; Theorem 3.3] that $\mathcal{S}(L_2(Q))$ is a Banach algebra. Furthermore, he obtained [4; Theorem 4.1] that the analytic Yeh–Feynman integral exists for every $[\tilde{\sigma}]$ in $S(L_2(Q))$, that is, for $\tilde{\sigma}$ given by (2.1)*

$$\int_{C_2(Q)}^{\text{any}\,\omega_\lambda} \tilde{\sigma}(x)dm(x) = \int_{L_2(Q)} \{\exp -\frac{1}{2\lambda}\|v\|_2^2\} d\sigma(v), \ \lambda \in \mathbf{C}^+,$$

and

$$\int_{C_2(Q)}^{\text{any}\,f_q} \tilde{\sigma}(x)dm(x) = \int_{L_2(Q)} \exp\{-\frac{i}{2q}\|v\|_2^2\} d\sigma(v) \tag{2.2}$$

for each real $q \neq 0$.

Next we give the information necessary for our discussion of the Banach algebra $\mathcal{F}(H)$ of Fresnel integrable functionals. The fundamental work on the space $\mathcal{F}(H)$ was done by Albeverio and Høegh–Krohn [1].

Let H be a separable infinite–dimensional Hilbert space over $\mathbf{R}$ (the real numbers) with inner product $<h_1, h_2>$ and norm $\|h\| = \sqrt{<h,h>}$. Let $M(H)$ denote the space of complex–valued, countably additive, Borel measures μ on H of finite total variation $\|\mu\|$. It is well known that $\mathcal{M}(H)$ is a commutative Banach algebra under the total variation norm, where convolution is taken as the multiplication.

DEFINITION 2.3. *The Fourier transform* $\hat{\mu}$ *of an element* μ *in* $\mathcal{M}(H)$ *is defined for all* k *in* H *by the formula*

$$\hat{\mu}(h) = \int_H \exp\{i <h, h_1>\} d\mu(h_1). \tag{2.3}$$

The Fresnel class $\mathcal{F}(H)$ *on* H *consists of functionals on* H *which are Fourier transforms of elements in* $\mathcal{M}(H)$. *We shall call* $\mathcal{F}(H)$ *the space of Fresnel integrable functionals on* H. *The Fresnel integral* $\mathcal{F}^{(q)}(\hat{\mu})$ *with parameter* q ($q \neq 0$, $\operatorname{Im}(q) \geq 0$) *is defined for all* $\hat{\mu}$ *in* $\mathcal{F}(H)$ *by the formula*

$$\mathcal{F}^{(q)}(\hat{\mu}) = \int_H \exp\{-\frac{i}{2q}\|h\|^2\} d\mu(h). \tag{2.4}$$

REMARK 2.2. *(1) The definition of* $\mathcal{F}^{(q)}(\hat{\mu})$ *makes sense : Continuity of the norm implies the functional* $\exp\{-\frac{i}{2q}\|\cdot\|\}$ *to be continuous; hence it is Borel measurable and at the same time bounded. If* $q = 1$, *then (2.4) is just the definition of the Fresnel integral given by Albeverio and Høegh–Krohn.*

(2) Define a mapping Φ *from* $\mathcal{M}(H)$ *to* $\mathcal{F}(H)$ *by the formula :* $\Phi(\mu) = \hat{\mu}$, $\mu \in \mathcal{M}(H)$. *Then* Φ *is injective, preserves addition and scalar multiplication, and converts convolution to pointwise multiplication. If we let* $\|\hat{\mu}\| = \|\mu\|$, *it follows immediately that* $(\mathcal{F}(H), \|\cdot\|)$ *is a commutative Banach algebra.*

Now we introduce a particular real separable Hilbert space H_2 which is appropriate for our purposes throughout the rest of this paper.

Let $\mathcal{H}_2$ be the space of real-valued functions γ on $Q = [0,a] \times [0,b]$ which are absolutely continuous and have the partial derivative $D_2\gamma = \frac{\partial^2\gamma}{\partial t \partial s}$ in $L_2(Q)$. Define

$$H_2 = \{\gamma \in \mathcal{H}_2 : \gamma(0,t) = \gamma(s,0) = 0,\ 0 \le s \le a,\ 0 \le t \le b\},$$

and define the inner product on H_2 as follows ;

$$(2.5) \qquad \begin{aligned} <\gamma_1,\gamma_2>_{H_2} &= <\gamma_1,\gamma_2> \\ &= \int_Q (D_2\gamma_1)(s,t)(D_2\gamma_2)(s,t)dsdt. \end{aligned}$$

Then H_2 becomes a separable Hilbert space over R.

Let $\mathcal{M}(H_2)$ be the space of all complex Borel measures on H_2. The Fourier transform $\hat{\mu}$ of μ in $\mathcal{M}(H_2)$ is defined for all γ in H_2 by the formula :

$$(2.6) \qquad \hat{\mu}(\gamma) = \int_{H_2} \exp\{i \int_Q (D_2\gamma)(s,t)(D_2\gamma_1)(s,t)dsdt\} d\mu(\gamma_1),$$

and the Fresnel integral $\mathcal{F}^{(q)}(\hat{\mu})$ on $\mathcal{F}(H_2)$ with parameter q ($q \neq 0, \mathrm{Im}(q) \ge 0$) is defined as follows ;

$$(2.7) \qquad \begin{aligned} \mathcal{F}^{(q)}(\hat{\mu}) &= \int_{H_2} \exp\{-\frac{i}{2q} \int_Q [(D_2\gamma)(s,t)]^2 ds\, dt\} d\mu(\gamma) \\ &= \int_{H_2} \exp\{-\frac{i}{2q} \|\gamma\|^2_{H_2}\} d\mu(\gamma),\ \hat{\mu} \in \mathcal{F}(H_2). \end{aligned}$$

For a detailed discussion of the absolutely continuous functions with two variables see [2,8].

3. THE RELATIONSHIP BETWEEN $S(L_2(Q))$ AND $\mathcal{F}(H_2)$

We begin this section, assuming that $\mathcal{H}_2$ and H_2 are identical with those in the previous section throughout this section and the next section.

DEFINITION 3.1. *The differentiation mapping* $D_2 : H_2 \to L_2(Q)$ *is defined by the formula*

$$D_2\gamma = \frac{\partial^2\gamma}{\partial t \partial s}, \quad \text{and} \tag{3.1}$$

the integration mapping $I_2 : L_2(Q) \to H_2$ *is defined by the formula*

$$(I_2 v)(s,t) = \int_0^t \int_0^s v(p,\gamma) dp d\gamma, \quad (s,t) \in Q. \tag{3.2}$$

The following lemma is quoted from [15; Theorem 4] without the proof.

LEMMA 3.1. *Let* v *be in* $L_2(Q)$ *and let* γ *be in* $\mathcal{H}_2$. *Then the P.W.Z. integral* $\int_Q v(s,t)\tilde{d}\gamma(s,t)$ *exists and we have*

$$\int_Q v(s,t)\tilde{d}\gamma(s,t) = \int_Q v(s,t)(D_2\gamma)(s,t) ds dt. \tag{3.3}$$

PROPOSITION 3.1. *The differentiation mapping* D_2 *is an isometric isomorphism of* H_2 *onto* $L_2(Q)$. *The integration mapping* I_2 *is the inverse of* D_2.

Proof. Given v in $L_2(Q)$, let $I_2 v$ be given by (3.2). Then $I_2 v$ is absolutely continuous, $(I_2 v)(0,t) = (I_2 v)(s,0) = 0$ for $0 \le s \le a$ and $0 \le t \le b$, and $D_2(I_2 v) = v$. Hence $I_2 v$ is in H_2 and D_2 maps H_2 onto $L_2(Q)$. It is easy to see that D_2 is injective and linear. The fact that D_2 preserves inner products is built into the definition (2.5) of the inner product on H_2.

DEFINITION 3.2. *Define two mappings* $\mathcal{D}_2 : \mathcal{M}(H_2) \to \mathcal{M}(L_2)$, *and* $\mathcal{I}_2 : \mathcal{M}(L_2) \to \mathcal{M}(H_2)$ *by the formulae :*

$$\mathcal{D}_2\mu = \mu \circ D_2^{-1}, \tag{3.4}$$

$$\mathcal{I}_2\sigma = \sigma \circ \mathcal{I}_2^{-1}. \tag{3.5}$$

PROPOSITION 3.2. *$\mathcal{D}_2$ is a Banach algebra isometric isomorphism of $\mathcal{M}(H_2)$ onto $\mathcal{M}(L_2)$, and $\mathcal{I}_2 = \mathcal{D}_2^{-1}$.*

Proof. We will carry out this proof as follows ;
Claim 1 ; $\mathcal{D}_2$ is bijective and linear.
This proof is routine.
Claim 2 ; $\mathcal{D}_2(\mu_1 * \mu_2) = \mathcal{D}_2(\mu_1) * \mathcal{D}_2(\mu_2)$ for $\mu_1, \mu_2 \in \mathcal{M}(H_2)$.
Let B be in the Borel class $\mathcal{B}(L_2)$ of $L_2 = L_2(Q)$. Using the change of Variables Theorem to justify the third equality, we obtain

$$\begin{aligned}
[\mathcal{D}_2(\mu_1) * \mathcal{D}_2(\mu_2)](B) &= \int_{L_2} (\mathcal{D}_2\mu_1)(B - v) d(\mathcal{D}_2\mu_2)(v) \\
&= \int_{L_2} \mu_1[D_2^{-1}(B - v)] d(\mu_2 \cdot D_2^{-1})(v) \\
&= \int_{H_2} \mu_1[D_2^{-1}B - \gamma] d\mu_2(\gamma) \\
&= (\mu_1 * \mu_2)(D_2^{-1}B) \\
&= [\mathcal{D}_2(\mu_1 * \mu_2)](B).
\end{aligned}$$

Claim 3 ; $\|\mathcal{D}_2\mu\| = \|\mu\|$ for every μ in $M(H_2)$.

Given a Hilbert space H_0, let $C_b(H_0)$ denote the space of complex–valued, bounded, continuous functionals on H_0. Then $C_b(H_0)$ is a Banach space under the supremum norm. Let $B_1(C_b(H_0))$ denote the unit ball of $C_b(H_0)$. It is clear that every element μ_1 of $\mathcal{M}(H_0)$ defines an element of the dual of $C_b(H_0)$ via integration. In fact, this imbedding of $\mathcal{M}(H_0)$ into the dual of $C_b(H_0)$ is a Banach space isometric isomorphism. Hence we obtain

$$\begin{aligned}
\|\mathcal{D}_2\mu\| &= \sup\{|\int_{L_2} f(v) d(\mathcal{D}_2\mu)(v)| : f \text{ is in } B_1(C_b(L_2))\} \\
&= \sup\{|\int_{L_2} f(v) d(\mu \circ D_2^{-1})(v)| : f \text{ is in } B_1(C_b(L_2))\} \\
&= \sup\{|\int_{H_2} f(D_2\gamma) d\mu(\gamma)| : f \text{ is in } B_1(C_b(L_2))\} \\
&\leq \|\mu\|
\end{aligned}$$

since $f \circ D_2$ is in $B_1(C_b(H_2))$ for every f in $B_1(C_b(L_2))$.

To get the opposite inequality, let σ in $\mathcal{M}(L_2)$ be such that $\mathcal{I}_2\sigma = \mu$ and then argue much as above that $\|\mathcal{I}_2\sigma\| \leq \|\sigma\|$; that is, $\|\mu\| \leq \|\mathcal{D}_2\mu\|$.

DEFINITION 3.3. *Let μ be in $M(H_2)$ and let $\sigma = \mathcal{D}_2\mu$. In addition to the functions $\hat{\mu}$ and $\tilde{\sigma}$, we may consider the ordinary Fourier transform $\hat{\sigma}$ of σ. $\hat{\sigma}$ is defined for every v in $L_2(Q)$ by the formula*

$$\hat{\sigma}(v) = \int_{L_2} \exp\{i \int_Q v(s,t)h(s,t)dsdt\}d\sigma(h).$$

The following theorem shows the relationships between $\hat{\mu}$, $\tilde{\sigma}$, and $\hat{\sigma}$ where μ is in $M(H_2)$, and σ is in $M(L_2)$.

THEOREM 3.1. *Let $\sigma = \mathcal{D}_2\mu$ where μ is in $M(H_2)$, and $\mathcal{D}_2$ is as in (3.4). Then*

(3.6) $\hat{\mu} = \hat{\sigma} \cdot D_2 = \tilde{\sigma}|_{H_2}$, *where $\tilde{\sigma}|_{H_2}$ is the restriction of $\tilde{\sigma}$ to H_2.*

Proof. Let γ be an arbitrary member of H_2. By Lemma 3.1 and the Change of Variables Theorem, we have

$$\begin{aligned}
\tilde{\sigma}(\gamma) &= \int_{L_2} \exp\{i \int_Q v(s,t)\tilde{d}\gamma(s,t)\}d\sigma(v) \\
&= \int_{L_2} \exp\{i \int_Q v(s,t)(D_2\gamma)(s,t)dsdt\}d\sigma(v) \\
&= \hat{\sigma}(D_2\gamma) \\
&= \int_{L_2} \exp\{i \int_Q v(s,t)(D_2\gamma)(s,t)dsdt\}d(\mu \circ D_2^{-1})(v) \\
&= \int_{H_2} \exp\{i \int_Q (D_2\gamma_1)(s,t)(D_2\gamma)(s,t)dsdt\}d\mu(\gamma_1) \\
&= \hat{\mu}(\gamma).
\end{aligned}$$

The correspondence between the Banach algebras $\mathcal{F}(H_2)$ and $\mathcal{S}(L_2(Q))$ can now be easily established by using the facts assembled above and certain fact already in the literature.

THEOREM 3.2. *The mapping* $\Phi : \mathcal{S}(L_2(Q)) \to \mathcal{F}(H_2)$ *defined by* $\Phi([\tilde{\sigma}]) = \tilde{\sigma}|_{H_2}$ *identifies* $\mathcal{S}(L_2(Q))$ *and* $\mathcal{F}(H_2)$ *isometrically and isomorphically as Banach algebras. The inverse of* Φ *is given by* $\Phi^{-1}(\hat{\mu}) = [\tilde{\mathcal{D}}_2\mu]$.

Proof. Let $[\tilde{\sigma}]$ be an arbitrary member in $\mathcal{S}(L_2(Q))$ and let $\mu = \mathcal{I}_2\sigma$ where $\mathcal{I}_2$ is given by (3.5). J.S. Chang has shown [4] that the mapping Φ_1, sending $[\tilde{\sigma}]$ to σ, is a Banach algebra isometric isomorphism of $\mathcal{S}(L_2(Q))$ onto $\mathcal{M}(L_2)$. By Proposition 3.2, $\mathcal{I}_2$, sending σ to $\mathcal{I}_2\sigma = \mu$, is an isometric isomorphism of $\mathcal{M}(L_2)$ onto $\mathcal{M}(H_2)$. Further the mapping Φ_2, sending $\mathcal{I}_2\sigma = \mu$ to $\hat{\mathcal{I}_2\sigma} = \hat{\mu}$, is an isometric isomorphism of $\mathcal{M}(H_2)$ onto $\mathcal{F}(H_2)$ as is discussed by Albeverio and Høegh–Krohn [1]. Hence $\Phi = \Phi_2 \circ \mathcal{I}_2 \circ \Phi_1$ is a Banach algebra isometric isomorphism of $\mathcal{S}(L_2(Q))$ onto $\mathcal{F}(H_2)$. Finally, by Theorem 3.1, the action of $\Phi([\tilde{\sigma}]) = \hat{\mu} = \hat{\mathcal{I}_2\sigma}$ on H_2 agrees with the action of $\tilde{\sigma}$ on H_2.

The following theorem shows that there is a simple relationship between the Fresnel integral of $\hat{\mu}$ and the analytic Yeh–Feynman integral of the corresponding element $\tilde{\sigma}$.

THEOREM 3.3. *Let* $\hat{\mu}$ *belong to* $\mathcal{F}(H_2)$ *and let* $[\tilde{\mathcal{D}}_2\mu]$ *be the corresponding element of* $\mathcal{S}(L_2(Q))$. *Let* q *be a non zero real parameter. Then the Fresnel integral* $\mathcal{F}^{(q)}(\hat{\mu})$ *of* $\hat{\mu}$ *is equal to the analytic Yeh–Feynman integral with parameter* q *of* $\tilde{\mathcal{D}}_2\mu$ *; that is,*

$$\int_{C_2(Q)}^{\text{any } f_q} (\tilde{\mathcal{D}}_2\mu)(x)\,dm(x) = \mathcal{F}^{(q)}(\hat{\mu}). \tag{3.7}$$

In fact, for any F *in* $[\tilde{\mathcal{D}}_2\mu]$,

$$\int_{C_2(Q)}^{\text{any } f_q} F(x)\,dm(x) = \mathcal{F}^{(q)}(\hat{\mu}).$$

Proof. By the equalities (2.2) and (2.4), and the Change of Variables

Theorem, we have

$$\begin{aligned}\int_{C_2(Q)}^{\text{any } f_q} (\tilde{\mathcal{D}_2\mu})(x)dm(x) &= \int_{L_2} \exp\{-\frac{i}{2q}\|v\|_2^2\}d(\mathcal{D}_2\mu)(v) \\ &= \int_{L_2} \exp\{-\frac{i}{2q}\|v\|_2^2\}d(\mu \circ D_2^{-1})(v) \\ &= \int_{H_2} \exp\{-\frac{i}{2q}\|D_2\gamma\|_2^2\}d\mu(\gamma) \\ &= \int_{H_2} \exp\{-\frac{i}{2q}\|\gamma\|_{H_2}^2\}d\mu(\gamma) \\ &= \mathcal{F}^{(q)}(\hat{\mu}).\end{aligned}$$

4. THE APPLICATIONS

In this section, we obtain a traslation theorem for our Fresnel integral $\mathcal{F}^{(q)}(\hat{\mu})$, and then, using this theorem and Theorem 3.3, we prove a Cameron–Martin type translation theorem for the analytic Yeh–Feynman integral. In order to proceed to our work, we will need two simple results about how certain measures are transformed by the mapping $\mathcal{D}_2 : \mathcal{M}(H_2) \to \mathcal{M}(L_2)$. These are slight modifications of Propositions 3 and 4 in [9] for our purpose. We omit the simple proofs.

PROPOSITION 4.1. *Let μ be in $\mathcal{M}(H_2)$ and let $\sigma = \mathcal{D}_2\mu$ be the corresponding element in $\mathcal{M}(L_2)$.*

(a) Let h be a complex–valued, Borel measurable functional on H_2. Then h is in $L_1(\mu)$ if and only if $h \circ I_2$ is in $L_1(\sigma)$. Further, if h is in $L_1(\mu)$ and μ_h is defined by $d\mu_h(\gamma) = h(\gamma)d\mu(\gamma)$, then μ_h is in $\mathcal{M}(H_2)$ and $\mathcal{D}_2(\mu_h) = \sigma_{h\circ I_2}$, where $d\sigma_{h\circ I_2}(v) = h(I_2(v))d\sigma(v)$.

(b) Let g be a complex–valued, Borel measurable functional on $L_2(Q)$. Then g is in $L_1(\sigma)$ if and only if $g \circ D_2$ is in $L_1(\mu)$. Further, if g is in $L_1(\sigma)$ and σ_g is defined by $d\sigma_g(v) = g(v)d\sigma(v)$, then σ_g is in $\mathcal{M}(L_2)$ and $\mathcal{I}_2(\sigma_g) = \mu_{g\circ D_2}$, $d\mu_{g\circ D_2}(\gamma) = g(D_2\gamma)d\mu(\gamma)$.

PROPOSITION 4.2. *Let μ be in $\mathcal{M}(H_2)$ and let $\sigma = \mathcal{D}_2\mu$ be the corresponding element in $\mathcal{M}(L_2)$.*

(a) Let R_H be an injective mapping from H_2 into H_2 which carries Borel sets to Borel sets. Then $\mu \circ R_H$ is in $\mathcal{M}(H_2)$ and $\mathcal{D}_2(\mu \circ R_H) = \sigma \circ R_L$ where $R_L = D_2 \circ R_H \circ I_2$.

(b) Let R_L be an injective mapping from $L_2(Q)$ into $L_2(Q)$ which carries Borel sets to Borel sets. Then $\sigma \circ R_L$ is in $\mathcal{M}(L_2)$ and $\mathcal{I}_2(\sigma \circ R_L) = \mu \circ R_H$, where $R_H = I_2 \circ R_L \circ D_2$.

NOTATION. *Given Z in H_2, let $T_z : C_2(Q) \to C_2(Q)$ be defined by $T_z(x) = x + z$. The restriction of T_z to H_2 carries H_2 into H_2.*

The following theorem is a slight modification of Theorem 5 in [9] which is called the translation theorem for the Fresnel integral.

THEOREM 4.1. *Let z be in H_2 and let $f = \hat{\mu}$ be in $\mathcal{F}(H_2)$. Let $f_z = f \cdot T_z$, and define $f^{(z)} : H_2 \to \mathbf{C}$ by $f^{(z)}(\gamma) = f(\gamma)\exp\{-i <\gamma, z>\}$. Let q be a non-zero real parameter. Then $f_z = \hat{\mu}_1$ and $f^{(qz)} = \hat{\mu}_2$ are in $\mathcal{F}(H_2)$, where μ_1 and μ_2 are defined by $d\mu_1(\gamma) = \exp\{i <\gamma, z>\}d\mu(\gamma)$, and $\mu_2 = \mu \circ T_{qz}$. Further, we have*

$$\mathcal{F}^{(q)}(f_z) = \exp\{\frac{iq}{2}\|z\|^2_{H_2}\}\mathcal{F}^{(q)}(f^{(qz)}). \tag{4.1}$$

Proof. By the definitions of f_z and $f^{(z)}$, and using Propositions 4.1 and 4.2, we have

$$\begin{aligned} f_z(\gamma) &= \int_{H_2} \exp\{i <\gamma, h>\}\exp\{i <z, h>\}d\mu(h) \\ &= \int_{H_2} \exp\{i <\gamma, h>\}d\mu_1(h) \\ &= \hat{\mu}_1(\gamma) \text{ for all } \gamma \text{ in } H_2, \end{aligned}$$

$$\begin{aligned} f^{(qz)}(\gamma) &= \exp\{-i < \gamma, qz>\}f(\gamma) \\ &= \int_{H_2} \exp\{i <\gamma, h>\}d\mu(h + qz) \\ &= \hat{\mu}_2(\gamma) \text{ for all } \gamma \text{ in } H_2. \end{aligned}$$

Then $f_z = \hat{\mu}_1$ and $f^{(qz)} = \hat{\mu}_2$ are in $\mathcal{F}(H_2)$. Finally, we obtain

$$\begin{aligned}\mathcal{F}^{(q)}(f_z) &= \int_{H_2} \exp\{-\frac{i}{2q}\|\gamma\|_{H_2}^2\}d\mu_1(\gamma)\\ &= \int_{H_2} \exp\{-\frac{i}{2q}\|\gamma\|_{H_2}^2\}\exp\{i <\gamma, z>\}d\mu(\gamma)\\ &= \exp\{\frac{iq}{2}\|z\|_{H_2}^2\}\int_{H_2} \exp\{-\frac{i}{2q}\|\gamma - qz\|_{H_2}^2\}d\mu(\gamma)\\ &= \exp\{\frac{iq}{2}\|z\|_{H_2}^2\}\int_{H_2} \exp\{-\frac{i}{2q}\|\gamma\|_{H_2}^2\}d\mu(\gamma + qz)\\ &= \exp\{\frac{iq}{2}\|z\|_{H_2}^2\}\mathcal{F}^{(q)}(f^{(qz)}).\end{aligned}$$

Now we will prove a translation theorem for the analytic Yeh–Feynman integrals of elements in $\mathcal{S}(L_2(Q))$, by using Theorems 3.3 and 4.1. This theorem has been obtained by Yoo lately (see [18; Theorem 5.2]).

THEOREM 4.2. *Let z be in H_2 and let F be in $[\tilde{\sigma}]$, where $[\tilde{\sigma}]$ is in $\mathcal{S}(L_2(Q))$. Let $F_z = F \circ T_z$ and define $F^{(z)} : C_2(Q) \to \mathbf{C}$ by $F^{(z)}(x) = F(x)\exp\{-i\int_Q (D_2 z)(s,t)\tilde{d}x(s,t)\}$. Let q be a non-zero real parameter. Then F_z belongs to $[\tilde{\sigma}_1]$ and $F^{(qz)}$ belongs to $[\tilde{\sigma}_2]$, where σ_1 and σ_2 are defined by $d\sigma_1(v) = \exp\{i\int_Q v(s,t)\tilde{d}z(s,t)\}d\sigma(v)$ and $\sigma_2 = \sigma \circ T_{D_2(qz)}$, respectively. Further, we have*

$$\int_{C_2(Q)}^{\text{any}\, f_q} F_z(x)dm(x) = \exp\{\frac{iq}{2}\|z\|_{H_2}^2\}\int_{C_2(Q)}^{\text{any}\, f_q} F^{(qz)}(x)dm(x). \tag{4.2}$$

Proof. Put $\mu = \mathcal{I}_2\sigma$ and $f = \hat{\mu}$, where σ is in $\mathcal{M}(L_2)$. let μ_1 and μ_2 be as in Theorem 4.1. Now put $\tau_1 = \mathcal{D}_2\mu_1$ and $\tau_2 = \mathcal{D}_2\mu_2$. Using Proposition 4.1 and Lemma 3.1, we have

$$\begin{aligned}d\tau_1(v) &= \exp\{i <z, I_2 v>_{H_2}\}d\sigma(v)\\ &= \exp\{i\int_Q (D_2 z)(s,t)(D_2 \circ I_2 v)(s,t)dsdt\}d\sigma(v)\\ &= \exp\{i\int_Q v(s,t)\tilde{d}z(s,t)\}d\sigma(v)\\ &= d\sigma_1(v).\end{aligned}$$

Using Proposition 4.2, we have

$$\tau_2 = \sigma \circ D_2 \circ T_{qz} \circ I_2.$$

Hence $\tau_2(B) = \sigma(B + D_2(qz)) = (\sigma \circ T_{D_2(qz)})(B) = \sigma_2(B)$ for every B in $\mathcal{B}(L_2)$, the Borel class of $L_2(Q)$. Thus we have shown that $\sigma_1 = \mathcal{D}_2\mu_1$ and $\sigma_2 = \mathcal{D}_2\mu_2$. Now we may rewrite the equality (4.1) as follws ;

$$\mathcal{F}^{(q)}(\hat{\mu}_1) = \exp\{\frac{iq}{2}\|z\|^2_{H_2}\}\mathcal{F}^{(q)}(\hat{\mu}_2). \tag{4.3}$$

It follows immediately from Theorem 3.3 that

$$\int^{\text{any}\, f_q}_{C_2(Q)} \tilde{\sigma}_1(x)dm(x) = \exp\{\frac{iq}{2}\|z\|^2_{H_2}\} \int^{\text{any}\, f_q}_{C_2(Q)} \tilde{\sigma}_2(x)dm(x). \tag{4.4}$$

Thus the theorem will be proved as soon as we show that F_z belongs to $[\tilde{\sigma}_1]$ and $F^{(qz)}$ belongs to $[\tilde{\sigma}_2]$. To do this, we proceed as follows ; We can easily show that

$$\begin{aligned}\tilde{\sigma}_1(x) &\approx \int_{L_2} \exp\{i \int_Q v(s,t)\tilde{d}x(s,t)\}d\sigma_1(v)\\ &\approx \int_{L_2} \exp\{i \int_Q v(s,t)\tilde{d}\,[x(s,t) + z(s,t)]\}d\sigma(v)\\ &\approx \tilde{\sigma}(x+z).\end{aligned}$$

The proof that $F_z \in [\tilde{\sigma}_1]$ will be finished if we show that $\tilde{\sigma}(x+z) = F(x+z)$ for s–a.e. x in $C_2(Q)$. Let $N = \{x \in C_2(Q) : F(x) \neq \tilde{\sigma}(x)\}$. Then N is scale–invariant null since $F \in [\tilde{\sigma}]$. Then $\tilde{\sigma}(x+z) = F(x+z)$ except for $x's$ such that x is in $N - z$. So it suffices to show that $\rho N - \rho z$ is Yeh–Wiener null for every positive real number ρ. But $m(\rho N) = 0$ since N is scale–invariant null. Also ρz is in H_2, and it is well known from the translation theory in Yeh–Wiener space [17] that translation by such elements preserves set of measure zero. Finally, we can show that

$F^{(qz)}$ belongs to $[\tilde{\sigma}_2]$ as follows ;

$$\begin{aligned}
\tilde{\sigma}_2(x) &\approx \int_{L_2} \exp\{i \int_Q v(s,t)\tilde{d}x(s,t)\} d\sigma_2(v) \\
&\approx \int_{L_2} \exp\{i \int_Q v(s,t)\tilde{d}x(s,t)\} d(\sigma \circ T_{D_2(qz)})(v) \\
&\approx \int_{L_2} \exp\{i \int_Q [v(s,t) - (D_2(qz))(s,t)]\tilde{d}x(s,t)\} d\sigma(v) \\
&\approx F(x) \exp\{-i \int_Q (D_2(qz))(s,t)\tilde{d}x(s,t)\} \\
&\approx F^{(qz)}(x).
\end{aligned}$$

Thus we have just proved that (4.2) holds.

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