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SMOOTH POINTS IN $C(X)$-SUBMODULES OF $C(X;B)$ *

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1. Introduction

Let X be a compact Hausdorff space and B a Banach space over either the real or complex number field $\mathbf{F}$. Let $C(X;B)$ denote the vector space of continuous functions from X into B. If $C(X;B)$ is equipped with the natural norm ($\|f\| = \sup\{\|f(x)\| : x \in X\}$ for any f in $C(X;B)$), then it is a Banach space. For any h in $C(X)$ (that is, $C(X;\mathbf{F})$) and f in $C(X;B)$, the pointwise product hf is in $C(X;B)$ and $C(X;B)$ can be considered as a so called Banach $C(X)$-module. Let W be any closed $C(X)$-submodule of $C(X;B)$. That is, W is a closed subspace satisfying $C(X)W = W$. This note is concerned with determining the smooth points in such a Banach space W.

For a general Banach space E, an element v in E is called a *smooth point* in E if there exists a unique v^* in E^* with $\|v^*\| = 1$ and $(v, v^*) = \|v\|$. It is well known that this is equivalent to v being a nonzero point of E where the norm is weakly (or Gateaux) differentiable. See Köthe's book [5], sections 26.3, 26.4 and 26.5 for a discussion of smooth points in general.

The smooth points of $C(X;B)$ are known and can be described as follows:

For a nonzero f in $C(X;B)$ to be a smooth point it is necessary and sufficient that there exists a point x_0 in X satisfying

(i) $\|f(x_0)\| > \|f(x)\|$, for all x in X, $x \neq x_0$

and

(ii) $f(x_0)$ is a smooth point in B.

For $B = R$, this result goes back to Banach and in the more general form has appeared in [1] and [6].

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In order to state the main theorem of this paper, for each x in X define $B(x) = \{f(x) : f \in W\}$, for a fixed closed $C(X)$–submodule W of $C(X;B)$. It turns out that each $B(x)$ is a closed subspace of B (Proposition 1 below) and $W = \{f \in C(X;B) : f(x) \in B(x)$, for all x in $X\}$.

THEOREM 1. *Let W be a closed $C(X)$–submodule of $C(X;B)$. Let f be a nonzero element of W. For f to be a smooth point of W it is necessary and sufficient that there exists a point x_0 in X satisfying*
(i) $\|f(x_0)\| > \|f(x)\|$, for all x in X, $x \neq x_0$ and
(ii) $f(x_0)$ is a smooth point in $B(x_0)$.

The proof of this theorem will be given in section 3 while some preliminary results will be presented in section 2. One of these results, theorem 2, identifying the extreme points of the unit ball in W^* may be of interest in its own right. An extension of the result to locally compact Hausdorff spaces X is given in section 4, and the paper concludes with some examples of $W's$ which arise somewhat naturally. It should be pointed out that Theorem 1 is only useful if the representation of W as a $C(X)$–submodule of some $C(X;B)$ is both natural and concrete.

2. Extreme Points in the Unit Ball of W^*

In this section, X is a compact Hausdorff space, B is a Banach space and W is a closed subspace of $C(X;B)$ satisfying $C(X)W = W$. Here, for h in $C(X)$ and f in $C(X;B)$, hf in $C(X;B)$ is defined by $(hf)(x) = h(x)f(x)$, for all x in X. For each x in X, let

$$B(x) = \{f(x) : f \in W\}.$$

PROPOSITION 1. *Each $B(x)$ is a closed subspace of B and for each b in $B(x)$ there exists f_b in W with $f_b(x) = b$ and $\|f_b\| = \|b\|$.*

Proof. Let b be an element of $\overline{B(x)}$. A Cauchy sequence $\{f_n\}$ in W will be constructed such that, for each natural number n,

$$\|f_n(x) - b\| < 2^{-n}.$$

If f in W is the limit of the $f_n's$, then $f(x) = b$. This will prove that $B(x)$ is closed in B and the function f will be modified to give the f_b as required. The construction of the sequence $\{f_n\}$ is inductive.

Let f_0 in W be such that $\|f_0(x) - b\| < 1$. Let $f_{-1} = f_0$. Suppose that $f_0, \ldots, f_n$ in W have been found so that $\|f_k(x) - b\| < 2^{-k}$ and $\|f_k - f_{k-1}\| < 3 \cdot 2^{-k}$, for $0 \leq k \leq n$. Let g_{n+1} in W be such that $\|g_{n+1}(x) - b\| < 2^{-(n+1)}$. Then $\|g_{n+1}(x) - f_n(x)\| < 3 \cdot 2^{-(n+1)}$. Let U be an open neighbourhood of x such that $\|g_{n+1}(y) - f_n(y)\| < 3 \cdot 2^{-(n+1)}$, for all y in U. Let V be an open set with $X \backslash U \subseteq V$ but x not in V. Let $\{h_1, h_2\}$ be a partition of unity on X subordinate to the cover $\{U, V\}$ of X. Let $f_{n+1} = h_1 g_{n+1} + h_2 f_n$. Then f_{n+1} satisfies the inductive requirements. Thus $\{f_n\}$ is a Cauchy sequence in W which has a limit f in W. Also $f(x) = b$.

It may very well be that $\|f\| > \|b\|$, but it is easy to modify f as follows. If $b = 0$, take $f_b = 0$. Otherwise, let U_1 be a neighbourhood of x with $f(y) \neq 0$ for all y in U_1. Let g from X into $[0,1]$ be continuous with support contained in U_1 satisfying $g(x) = 1$ and define h in $C(X)$ by, $h(y) = (\|b\|/\|f(y)\|)g(y)$ if y is in U_1 and $h(y) = 0$ otherwise. Then $f_b = hf$ is an element of W with $f_b(x) = b$ and $\|f_b\| = b$.

Remark. The fact that each $B(x)$ is closed is known and holds in much greater generality then the situation at hand (for example, see the appendix of [2], where a strong result of Douady and dal Soglio–Hérault on the existence of cross–sections of Banach bundles is presented). The result given above in Proposition 1 is really a highly simplified version of this more general result.

PROPOSITION 2. *$W = \{f \in C(X;B) : f(x) \in B(x)$, for all x in $X\}$.*

Proof. Let $Z = \{f \in C(X;B) : f(x) \in B(x)$, for all x in $X\}$. Clearly $W \subseteq Z$. Let f be an element of Z and $\varepsilon > 0$. For each x in X, there is a g_x in W with $g_x(x) = f(x)$, by Proposition 1. Let $U(x)$ be a neighbourhood of x such that $\|g_x(y) - f(y)\| < \varepsilon$, for all y in $U(x)$. Since X is compact, there exist $x_1, \ldots, x_n$ in X such that $X \subset \cup_{i=1}^{n} U(x_i)$.

Let $\{h_1, \ldots, h_n\}$ be a partition of unity on X, subordinate to the cover $\{U(x_1), \ldots, U(x_n)\}$. Then the function $g = \sum_{i=1}^{n} h_i g_{x_i}$ is an element of

W with $\|g - f\| < \varepsilon$. Since W is closed, it must be that $Z = W$.

The above propositions clarify the structure of the kinds of subspace W which are under consideration and will now be used to study W^*, the dual space of W.

DEFINITION. *Let ρ be an element of W^*. A point x in X is called a support point of ρ if, for any neighbourhood U of x, there exists an f in W such that f is zero on $X \backslash U$ and $(f, \rho) \neq 0$. Let $S(\rho)$ denote the set of support points of ρ.*

Clearly $S(\rho)$ is a closed subset of X. By using a simple partition of unity argument on the compact space X it can be shown that $S(\rho)$ is nonempty if ρ is a nonzero element of W^*.

For any Banach space E, let $b_1(E)$ denote the unit ball of E. If K is a convex set in E, let $\text{ext}(K)$ denote the set of extreme points of K.

LEMMA 1. *If ρ is an element of $\text{ext}(b_1(W^*))$, then $S(\rho)$ is a singleton set.*

Proof. Suppose, to the contrary that there are two distinct points x and y in $S(\rho)$. The idea of the proof is to find nonzero ρ_1 and ρ_2 in W^* with $\rho = \rho_1 + \rho_2$ and $1 = \|\rho\| = \|\rho_1\| + \|\rho_2\|$. This will lead to a contradiction of ρ being an extreme point of $b_1(W^*)$.

To construct ρ_1 and ρ_2, let U be an open neighbourhood of x with y in $X \backslash \overline{U}$. Consider D, the collection of all closed subsets K of X with $K \subseteq U$. Order D by: $K_1 \leq K_2$ if $K_1 \subseteq K_2$, for K_1 and K_2 in D. For each K in D, fix h_k, a continuous function from X to $[0,1]$ such that

$$h_K(z) = \begin{cases} 1 & \text{if } z \in K \\ 0 & \text{if } z \in X \backslash U. \end{cases}$$

For any h in $C(X)$ and ϕ in W^* define $h\phi$ in W^* by

$$(f, h\phi) = (hf, \phi), \quad \text{for all} \quad f \quad \text{in} \quad W.$$

Then $\|h_K \rho\| \leq \|h_K\|_\infty \|\rho\| \leq \|\rho\| = 1$, for each K in D. Now D is a directed set and $\{h_K \rho : K \in D\}$ is a net in the weak* –compact set $b_1(W^*)$. So there exists a subnet of $\{h_K \rho : K \in D\}$ which converges (weak*) to some ρ_1 in $b_1(W^*)$. By taking a further subnet if necessary, it can be assumed that there is a dircted set Δ and subnets $\{h_\alpha\rho\}_{\alpha\in\Delta}$ and $\{(1-h_\alpha)\rho\}_{\alpha\in\Delta}$ of $\{h_K\rho\}$ and $\{(1-h_K)\rho\}$, respectively, such that $h_\alpha\rho$ converges weak* to ρ_1 and $(1-h_\alpha)\rho$ converges weak* to ρ_2, say, in $b_1(W^*)$. By uniqueness of limit points $\rho=\rho_1+\rho_2$.

It can be checked easily that,

$\|\rho_1\|=\sup\{|(f,\rho_1)| : f\in b_1(W) \text{ with support}(f)\subseteq U\}$ and, for any closed set $K\subseteq U$,

$$\|\rho_2\|=\sup\{|(f,\rho_2)| : f\in b_1(W) \text{ with support}(f)\subset X\backslash K\}.$$

So for any $\varepsilon>0$, there exists an f in $b_1(W)$ such that support$(f)\subseteq U$ and $(f,\rho_1)>\|\rho_1\|-\varepsilon/3$. Also, there exists g in $b_1(W)$ with support$(g)\subseteq X\backslash$support(f), such that $(g,\rho_2)>\|\rho_2\|-\varepsilon/3$. Now $f+zg$ is in $b_1(W)$ for any z in F with $|z|=1$ so it must be that $|(g,\rho_1)|<\varepsilon/3$, otherwise one could contradict $(f,\rho_1)>\|\rho_1\|-\varepsilon/3$.

It is clear, since support$(f)\subseteq U$, that $(f,\rho)=(f,\rho_1)$. Therefore,

$$\|\rho_1\|+\|\rho_2\|-\varepsilon<(f,\rho_1)-|(g,\rho_1)|+(g,\rho_2)\le|(f+g,\rho)|\le\|\rho\|.$$

Since ε is arbitrary and $\|\rho\|\le\|\rho_1\|+\|\rho_2\|$, it has been shown that $\|\rho\|=\|\rho_1\|+\|\rho_2\|$.

Neither ρ_1 nor ρ_2 is 0 since x is in $S(\rho_1)$ and y is in $S(\rho_2)$. So

$$\rho=\|\rho_1\|(\rho_1/\|\rho_1\|)+\|\rho_2\|(\rho_2/\|\rho_2\|),$$

which contradicts the hypothesis that ρ is an extreme point of $b_1(W^*)$. Therefore, $S(\rho)$ is a singleton set.

LEMMA 2. *Let ρ be an element of W^* with $S(\rho)=\{x\}$ for some x in X. Then, for any fixed neighbourhood U of x,*

$$\|\rho\|=\sup\{|(f,\rho)| : f\in b_1(W) \text{ and support}(f)\subseteq U\}.$$

Proof. Using the open set U, one can construct ρ_1 and ρ_2 as in the proof of lemma 1 and show that $\rho_2=0$. Then $\rho=\rho_1$ and the conclusion follows easily as for ρ_1 in lemma 1.

LEMMA 3. *Let ρ be an element of W^* with $S(\rho) = \{x\}$ for some x in X. If f and g are in W with $f(x) = g(x)$, then $(f, \rho) = (g, \rho)$.*

Proof. Let $\varepsilon > 0$ be given and let U be a neighbourhood of x such that $\|f(y) - g(y)\| < \varepsilon$ for all y in U. Let V be a neighbourhood of x with $\overline{V} \subseteq U$ and let h be a continuous function from X into $[0, 1]$ with $h(y) = 1$, for all y in V and $h(y) = 0$, for all y in $X \backslash U$. Then,

$$|(f - g, \rho)| \leq |((1 - h)(f - g), \rho)| + |(h(f - g), \rho)| \leq 0 + \varepsilon\|\rho\|,$$

using lemma 2 and the fact that ρ has no points of support in $X \backslash V$ where $(1 - h)(f - g)$ is supported.

Since ε is arbitrary, it must be that $(f, \rho) = (g, \rho)$.

DEFINITION. *Let x be an element of X and F an element of $B(x)^*$. Define a linear functional $e[F, x]$ on W by,*

$$(f, e[F, x]) = (f(x), F), \quad \textit{for all} \quad f \quad \textit{in} \quad W.$$

It is clear that $e[F, x]$ is in W^* and $\|e[F, x]\| = \|F\|$.

PROPOSITION 3. *Let ρ be an element of W^* with $S(\rho) = \{x\}$, for some x in X. Then there is an F in $B(x)^*$ with $\rho = e[F, x]$.*

Proof. For each b in $B(x)$, choose f_b in W with $f_b(x) = b$ and $\|f_b\| = \|b\|$, using Proposition 1. Define F from $B(x)$ into F, by $(b, F) = (f_b, \rho)$, for all b in $B(x)$. This is independent of the choice of f_b by lemma 3. Similarly, lemma 3 is involved to show that F is linear. Moreover,

$|(b, F)| = |(f_b, \rho)| \leq \|b\|\,\|\rho\|$, for all b in $B(x)$. Hence, F is in $B(X)^*$. Finally, for any f in W, $f(x) = f_{f(x)}(x)$, so

$$(f, \rho) = (f_{f(x)}, \rho) = (f(x), F) = (f, e[F, x]). \text{ Thus } \rho = e[F, x].$$

Remark. It is clear that if F is a nonzero element of $B(x)^*$, then $S(e[F, x]) = \{x\}$.

All of the necessary preliminaries to the main result of this section have now been obtained. The characterization of the extreme points in the dual unit ball is a prerequisite for the identification of smooth points. The theorem below generalizes the well known result for $C(X;B)$.

THEOREM 2. *Let X be a compact Hausdorff space and B a Banach space. Let W be a closed $C(X)$–submodule of $C(X;B)$. Then*

$$\operatorname{ext} b_1(W^*) = \{e[F,x] : x \in X \text{ and } F \in \operatorname{ext} b_1(B(x)^*)\}.$$

Proof. Let ρ be in ext $b_1(W^*)$. By lemma 1, $S(\rho) = \{x\}$, for some x in X. By proposition 3, there exists an F in $B(X)^*$ such that $\rho = e[F,x]$. Since $\|F\| = \|\rho\| = 1$, F is in $b_1(B(x)^*)$. If F_1 and F_2 are in $b_1(B(x)^*)$ with $F = (F_1 + F_2)/2$, then

$$\rho = e[F,x] = (e[F_1,x] + e[F_2,x])/2.$$

Thus, $e[F_1,x] = e[F_2,x]$, which, by proposition 1, implies that $F_1 = F_2$. Therefore, F is an extreme point in $b_1(B(x)^*)$.

Conversely, suppose F is in ext $b_1(B(x)^*)$ and $\rho = e[F,x]$, for some x in X. If ρ_1 and ρ_2 are any two elements of $b_1(W^*)$ with $\rho = (\rho_1 + \rho_2)/2$, then it will be shown that $\rho_1 = \rho_2 = \rho$ which proves that ρ is an extreme point of $b_1(W^*)$. The first step in showing this is to prove $S(\rho_i) = \{x\}$ for $i = 1, 2$.

To see that $S(\rho_i) = \{x\}$ for each i, suppose that a point $y \neq x$ is in the support of ρ_1, say. Let U be a neighbourhood of x with y in $X \backslash \overline{U}$. As in the proof of lemma 1, ρ, ρ_1 and ρ_2, can be cut down to functionals supported on $\overline{U}$, call then ρ_U, ρ_{1U} and ρ_{2U}, respectively. Of course, $\rho_U = \rho$ but $\|\rho_{1U}\| < \rho_1\|$, so

$$1 = \|\rho\| = \|\rho_U\| = \|(\rho_{1U} + \rho_{2U})/2\| \leq (\|\rho_{1U}\| + \|\rho_{2U}\|)/2 < 1,$$

which is a contradiction. Thus, $S(\rho_1) = \{x\}$. Likewise, $S(\rho_2) = \{x\}$.

Now, by proposition 3, there exist F_1 and F_2 in $b_1(B(x)^*)$ such that $\rho_i = e[F_i,x]$ for $i = 1, 2$. This implies $F = (F_1 + F_2)/2$. So $F_1 = F_2$ and $\rho_1 = \rho_2$.

3. The Proof of Theorem 1

Suppose f is a smooth point in W. It may be assumed that $\|f\| = 1$. Since X is compact, there exists a point x_0 in X such that $\|f(x_0)\| = 1$.

This point must be unique, for if y is another point of X with $\|f(y)\| = 1$, then the Hahn–Banach theorem yields F_y in $b_1(B(y)^*)$ and F_{x_0} in $b_1(B(x_0)^*)$ with $(f(x_0), F_{x_0}) = (f(x), F_{x_0}) = 1$. That is, $(f, e[F_y, y]) = (f, e[F_{x_0}, x_0]) = 1$. Clearly, if $x_0 \neq y$, then $e[F_{x_0}, x_0] \neq e[F_y, y]$ which contradicts F being a smooth point in W. This proves that (i) of theorem 1 must hold.

To show that (ii) holds, let F_1 and F_2 be two elements in $b_1(B(x_0)^*)$ such that $(f(x_0), F_1) = (f(x_0), F_2) = 1$. Then

$(f, e[F_1, x_0]) = (f, e[F_2, x_0]) = 1$, which implies $e[F_1, x_0] = e[F_2, x_0]$ and again, by proposition 1, $F_1 = F_2$. Thus, $f(x_0)$ is a smooth point of $B(x_0)$.

Conversely, let f be an element of W which satisfies (i) and (ii) and assume without loss of generality that $\|f\| = 1$. Consider the set of support functionals to $b_1(W)$ at f, $Q = \{\rho \text{ in } b_1(W^*) : (f, \rho) = 1\}$. Then Q is a nonempty weak*–closed convex subset of $b_1(W^*)$ and $\operatorname{ext} Q \subseteq \operatorname{ext}(b_1(W^*))$. Let ρ be an extreme point of Q. By theorem 2, there exists x in X and F in $\operatorname{ext} b_1(B(x)^*)$ such that $\rho = e[F, x]$. Then

$$1 = (f, \rho) = (f(x), F) \leq \|f(x)\| \, \|F\| \leq 1.$$

So $\|f(x)\| = 1$, which implies $x = x_0$, since f satisfies (i). Moreover, F is uniquely detemined as a support functional to the smooth point $f(x_0)$ in $B(x_0)$. Thus, ext Q is a singleton, which implies that Q is a singleton by the Krein–Milman theorem. Thus, f is a smooth point in W. This completes the proof of theorem 1.

4. Extension to locally compact X

In this section, let X be a locally compact Hausdorff space and let $C(X)$ and $C(X; B)$ denote the Banach spaces of bounded continuous functions on X with values in the scalars and B, respectively. Let W be a closed $C(X)$-submodule of $C(X; B)$ and define $B(x)$, for each x in X as before.

The smooth points of $C(X; B)$ were identified in [7] and that result can be extended to W by using theorem 1 of this paper and making the appropriate notational changes in the proof of theorem 3.1 of [7]. The details are left to the interested reader.

THEOREM 3. *Let X be a locally compact Hausdorff space and B a Banach space. Let W be a closed $C(X)$–submodule of $C(X;B)$. An element f of W is a smooth point of W if only if*

(i) there exists an x_0 in X with $\|f(x_0)\| > \|f(x)\|$, for all x in X, $x \neq x_0$,

(ii) there exists a compact set K with x_0 in the interior K^0 of K such that

$$\sup\{\|f(y)\| : y \in X \backslash K^0\} < \|f\|,$$

and

(iii) $f(x_0)$ is a smooth point of $B(x_0)$.

5. Examples

The first example arises if one considers the space of kernel functions for certain Volterra type integral operators.

Let $X = [0,1]$ and let p and q be elements of $[1,\infty]$ such that $1/p + 1/q = 1$. For any k mapping the triangle $\{(x,y) : 0 \leq y \leq x \leq 1\}$ into F satisfying,

(a) for each x in $[0,1]$, $k(x,\cdot)$ is in $L^p[0,x] \subseteq L^p[0,1]$, and

(b) the map $x \to k(x,\cdot)$ is continuous from $[0,1]$ to $L^p[0,1]$, the operator K, defined by

$Kf(x) = \int_0^x k(x,y)f(y)dy$, for x in $[0,1]$ and each f in $L^p[0,1]$, is a bounded linear operator from $L^q[0,1]$ into $C[0,1]$ with $\|K\| = \sup\{\|k(x,\cdot)\|_p : 0 \leq x \leq 1\}$.

Denote ty WV_p the Banach space of all functions k satisfying (a) and (b) above with $\|k\| = \sup\{\|k(x,\cdot)\|_p : 0 \leq x \leq 1\}$. Clearly WV_p can be identified with the $C[0,1]$–submodule of $C([0,1];L^p[0,1])$ consisting of functions whose value at x in $[0,1]$ is an element of $L^p[0,1]$ which vanishes almost everywhere on $(x,1]$. With $B = L^p[0,1]$, $B(x) = L^p[0,x]$ for $0 \leq x \leq 1$. It is well known that for $p = 1$, the points of smoothness in $L^1[a,b]$ are those functions which are nonzero almost everywhere. Any nonzero element of $L^p[a,b]$ is a smooth point if $1 < p < \infty$ and $L^\infty[a,b]$ has no smooth points. Thus, for the space WV_p there are three different characterizations of smooth points depending on p.

1. ($p = 1$) For f in WV_1 to be a smooth point it is necessary and sufficient that there exists a unique x_0 in $[0,1]$ with $\|f(x_0)\|_1 = \|f\|$ and $[f(x_0)](y) \neq 0$ for almost all y in $[0,x_0]$.

2. ($1 < p < \infty$) For f in WV_p to be a smooth point it is necessary and sufficient that there exists a unique x_0 in $[0,1]$ with $\|f(x_0)\|_p = \|f\|$.
3. ($p = \infty$) WV_∞ has no smooth points.

The next example is a C^*–algebra which is closely related to the group C^*–algebra of the group of rigid motions of the plane. Before considering this example note that the smooth points in the sapce $K(H)$ of compact operators on a Hilbert space H are known (see [3] or [4]). Perhaps the most useful characterization of smooth points in $K(H)$ is the following:

> For T in $K(H)$ to be a smooth point it is necessary and sufficient that the maximal eigenvalue of T^*T have multiplicity 1.

This can easily be derived from the results of [3] or [4].

Let $X = [0,1]$ and let H be a separable infinite dimensional Hilbert space with a fixed orthonormal basis. Let WC denote the subspace of $C([0,1];K(H))$ consisting of all functions f in $C([0,1];K(H))$ which vanish at 1 and for which $f(0)$ is diagonal with respect to the fixed basis in H. Then WC has a natural C^*–algebra structure and, although it is by no means typical, many C^*–algebras have a similar description.

For an f in WC to be a smooth point it is necessary and sufficient that f achieves its norm at a unique x_0 in $[0,1)$ and $[f(x_0)]^*[f(x_0)]$ has its maximal eigenvalue with multiplicity 1. Note, if $x_0 = 0$, then $f(0)$ should be a smooth point in the compact diagonal operators with respect to the fixed basis. But this space is isomorphic to c_0, in a natural way, where the smooth points are well known. In fact, an operator $T = \mathrm{diag}(a_1, a_2, a_3, \dots)$ is a smooth point in $B(0)$ if and only if there exists a unique n_0 such that $|a_{n_0}| = \max\{|a_n| : n = 1,2,3,\dots\}$, which holds if and only if T^*T has its maximal eigenvalue with multiplicity 1.

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