Comm. Korean Math. Soc. 4 (1989), No. 2, pp. 359~365

MAXIMAL INEQUALITIES WITH RESPECT TO PRODUCT MEASURES *

KYUNG-HWA KIM

1. Introduction

Let μ be a positive Borel measure on $\mathbf{R}^n$ such that $\mu(K) < \infty$ for any compact set $K \subset \mathbf{R}^n$. We assume that $\mu(I) > 0$ for any non-void interval I in $\mathbf{R}^n$. We note that μ can be a weight i.e., $d\mu(x) = w(x)dx$, where $w > 0$ a.e. on $\mathbf{R}^n$ and $w \in L^1_{\text{loc}}(\mathbf{R}^n)$.

We assume that all measures appeared in this paper have the above properties.

Consider the following weighted maximal functions.

$$M_Q^* f(x) = \sup_{Q(x)} \frac{1}{\mu(Q(x))} \int_{Q(x)} |f(y)| d\mu(y),$$

where the sup is taken over all cubes $Q(x)$, centered at x whose edges are parallel to the coordinate axes. From now on, cubes will always mean the ones with edges parallel to coordinate axes.

We can show, using the Besicovitch covering lemma, that M_Q^* is of weak-type $(1,1)$ and hence by the Marcinkiewicz interpolation theorem M_Q^* is of type (p,p) for $p > 1$. That is, we have

THEOREM A. *i) There is a constant $C > 0$ such that*

$$\mu\{x : M_Q^* f(x) > \alpha\} \le \frac{C}{\alpha} \int |f| d\mu, \quad \text{for any} \quad \alpha > 0$$

and for any $f \in L^1(d\mu)$.

Received May 20 1989.
*이 논문은 이화여자대학교 교수연구기금의 지원으로 이루어지었음.

(ii) For $1 < p \leq \infty$, $\|M_Q^* f\|_{p,\mu} \leq C\|f\|_{p,\mu}$, *where* C *is a constant independent of* f.

Proof. For the proof we refer to [4].

Now if we consider

$$M_Q f(x) = \sup_{x \in Q} \frac{1}{\mu(Q)} \int_Q |f(y)| d\mu(y),$$

where the sup is taken over all cubes containing x, then we need a doubling condition for μ to ensure the boundedness of M_Q. We will show

LEMMA 1. *If* μ *satisfies a doubling condition, i.e., there is a constant* $C > 0$ *such that* $\mu(Q^*) \leq C\mu(Q)$, *where* Q^* *is the cube concentric with* Q *and twice the size of* Q, *then we have*

i) $\mu\{x : M_Q f(x) > \alpha\} \leq \frac{C}{\alpha} \int |f| d\mu$, *for some constant* C *independent of* f *and* α ;

ii) For $1 < p \leq \infty$, *there is a constant* C *such that* $\|M_Q f\|_{p,\mu} \leq C\|f\|_{p,\mu}$.

But if we consider the strong maximal function

$$M_R f(x) = \sup_{x \in R} \frac{1}{\mu(R)} \int_R |f(y)| d\mu(y),$$

where the sup is taken over all intervals R *containing* x *in* $\mathbf{R}^n$, *then* M_R *does not have the boundedness as above, in general. To have the boundedness of* M_R *we need some condition on the measure* μ. *We have the following two theorems by R. Fefferman and Capri–Fava.*

THEOREM B (R. FEFFERMAN, [4]). *Suppose on* $\mathbf{R}^n$, $d\mu(x) = w(x)dx$, *where* w *is a function which has the property of being uniformly in the class* A_∞ *in each variable separately. Then* M_R *is a bounded operator on* $L^p(d\mu)$ *for* $1 < p \leq \infty$.

For the definition of A_∞ and the proof we refer to [4].

THEOREM C (CAPRI AND FAVA, [1]). *Let* $\mu = \mu_1 \times \cdots \times \mu_n$ *be the product of* n *Radon measures* μ_i *of* $\mathbf{R}^1$. *Then there exists a constant*

C depending only on the dimension n such that for every positive λ we have the inequality

$$\mu\{x : M_R f(x) > 4\lambda\} \leq C \int \frac{|f|}{\lambda} (log^+ \frac{|f|}{\lambda})^{n-1} d\mu.$$

In this paper we show the boundedness of M_R without the regularity of the measures as in [1] but with the dobuling condition. That is, as our main theorem, we show

THEOREM. *Let μ_i, $i = 1, \ldots, n$, be one dimensional measures and $\mu = \mu_1 \times \cdots \times \mu_n$. Suppose that each μ_i satisfies the doubling condition. Then we have*

i) $\|M_R f\|_{p,\mu} \leq C \|f\|_{p,\mu}$ *for* $1 < p \leq \infty$
ii) $\mu\{x : M_R f(x) > \lambda\} \leq C \int \frac{|f|}{\lambda} (1 + log^+ \frac{|f|}{\lambda})^{n-1} d\mu$ *for* $\lambda > 0$.

This generalizes a classical result by Jessen, Marcinkiewicz and Zygmund [6].

2. Maximal inequalities

We first show

LEMMA 1. *If μ satisfies the doubling condition, i.e., there is a constant $C > 0$ such that $\mu(Q^*) \leq C\mu(Q)$, where Q^* is the cube concentric with Q and twice the size of Q, then we have*

i) $\mu\{x : M_Q f(x) > \alpha\} \leq \frac{C}{\alpha} \int |f| d\mu$, for some constant C indepencent of f and α;

ii) For $1 < p \leq \infty$, there is a constant C such that $\|M_Q f\|_{p,\mu} \leq C\|f\|_{p,\mu}$.

Proof. For any cube Q containing x let Q' be the smallest cube centered at x and containing Q. Then from the doubling condition we can easily see that there is a constant C which depends on the constant in the doubling condition such that $\mu(Q') \le C\mu(Q)$. Therefore,

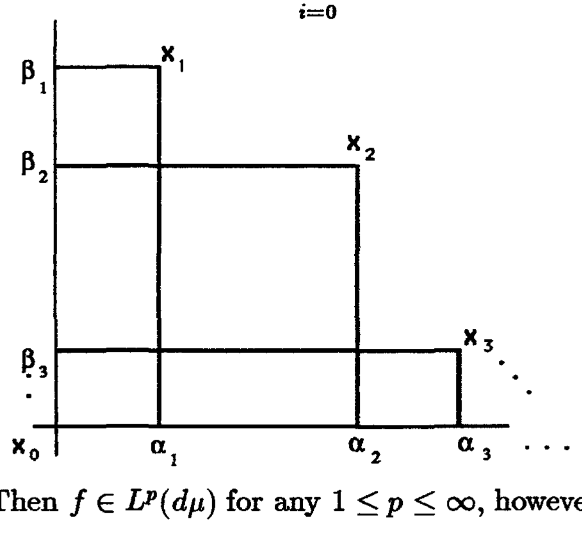
$$\frac{1}{\mu(Q)}\int_Q |f(y)|d\mu(y) \le \frac{1}{\mu(Q)}\int_{Q'} |f(y)|d\mu(y) \le \frac{C}{\mu(Q')}\int_{Q'} |f(y)|d\mu(y).$$

Hence, $M_Q f(x) \le C M_Q^* f(x)$ and the lemma follows from Theorem A.

Remark. Part ii) was appeared in [2] but not part i). We give this lemma here because it is essential for the proof of our main theorem.

The following example given in [4] shows that we need some condition on the measure to have the boundedness of the strong maximal function M_R.

Example [4]. Let the numbers α_i be positive and increasing and the β_i positive and decreasing. Set $x_i = (\alpha_i, \beta_i) \in \mathbf{R}^2$ and $R_i = [0, \alpha_i] \times [0, \beta_i] \subset \mathbf{R}^2$. Define $d\mu = \sum_{i=0}^{\infty} \delta_{x_i}$, where $x_0 = (0,0)$ and let $f = \chi_{R_1}$.



Then $f \in L^p(d\mu)$ for any $1 \le p \le \infty$, however,

$$M_R f(x_i) \ge \frac{1}{\mu(R_i)}\int_{R_i} f\,d\mu = \frac{1}{2} \quad \text{for} \quad i \ge 2.$$

Therefore, $\{x_2, x_3, \dots\} \subset \{M_R f(x) > \alpha\}$ for $0 < \alpha < \frac{1}{2}$ and since $\mu\{x_2, x_3, \dots\} = \infty$, we cannot have the weak–type (p,p) inequality for any $1 \le p < \infty$.

Now for $i = 1, \dots, n$, let μ_i be a one–dimensional measure with the doubling condition and let $\mu = \mu_1 \times \cdots \times \mu_n$. Define, for $i = 1, \dots, n$,

$$\begin{aligned} & M_i f(x) \\ &= M_i f(x_1, \dots, x_n) \\ &= \sup_{x_i \in I} \frac{1}{\mu_i(I)} \int_I |f(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)| d\mu_i(t), \end{aligned}$$

where the sup is taken over all intervals I in $\mathbf{R}^1$ containing the i^{th} coordinate x_i of x. Then each M_i is measurable in $\mathbf{R}^n$ and

$$M_R f \le M_n \cdots M_1 f.$$

LEMMA 2. *Suppose each μ_i and μ be as above. Then each M_i satisfies*

i) $\mu\{x \in \mathbf{R}^n : M_i f(x) > \lambda\} \le \frac{C_i}{\lambda} \int_{\mathbf{R}^n} |f| d\mu$ for some constant C_i depending only on the measure μ_i and for all $\lambda > 0$.

ii) $\|M_i f\|_{p,\mu} \le C_i' \|f\|_{p,\mu}$ for $1 < p \le \infty$, where C_i' is a constant depending on μ_i and p.

Proof. Since $\|M_i f\|_{p,\mu} \le \|f\|_{p,\mu}$ is obvious, by Marcinkiewicz interpolation theorem, it's enough to show i).

For each $i = 1, \dots, n$, since μ_i satisfies the doubling condition, by lemma 1,

$$\begin{aligned} & \mu_i\{x_i : M_i f(x_1, \dots, x_i, \dots, x_n) > \lambda\} \\ & \le \frac{C_i}{\lambda} \int_{\mathbf{R}^1} |f(x_1, \dots, x_i, \dots, x_n)| d\mu_i(x_i) \end{aligned}$$

where C_i is a constant depending on the measure μ_i.

Therefore, by Fubini,

$$
\begin{aligned}
&\mu\{x \in \mathbf{R}^n : M_i f(x) > \lambda\} \\
&= \int \cdots \int \mu_i\{x_i : M_i f(x_1, \ldots, x_i, \ldots, x_n) > \lambda\} d\mu_1(x_1) \ldots \\
&\qquad d\mu_{i-1}(x_{i-1}) d\mu_{i+1}(x_{i+1}) \ldots d\mu_n(x_n) \\
&\leq \int \cdots \int \frac{C_i}{\lambda} \int_{\mathbf{R}^1} |f(x_1, \ldots, x_i, \ldots, x_n)| d\mu_i(x_i) d\mu_1(x_1) \ldots d\mu_n(x_n) \\
&= \frac{C_i}{\lambda} \int_{\mathbf{R}^n} |f| d\mu.
\end{aligned}
$$

Now, we have our main theorem.

THEOREM. *Let μ_i, $i = 1, \ldots, n$, be one dimensional measures and $\mu = \mu_1 \times \cdots \times \mu_n$. Suppose that each μ_i satisfies the doubling condition. Then we have*

i) $\|M_R f\|_{p,\mu} \leq C \|f\|_{p,\mu}$ *for* $1 < p \leq \infty$

ii) $\mu\{x : M_R f(x) > \lambda\} \leq C \int \frac{|f|}{\lambda}(1 + log^+ \frac{|f|}{\lambda})^{n-1} d\mu$ *for* $\lambda > 0$.

Proof. Since $M_R f \leq M_n \cdots M_1 f$, by repeated use of part ii) of lemma 2, we get i).

From lemma 2, each M_i is a maximal operator in the sense of the definition given in [3]. Therefore, by theorem 1 in [3]

$$
\mu\{x : M_n \cdots M_1 f > \lambda\} \leq C \int \frac{|f|}{\lambda}(1 + \log^+ \frac{|f|}{\lambda})^{n-1} d\mu
$$

and so we have ii).

Remark. We do not get a weak–type $(1,1)$ inequality for the strong maximal function M_R even when μ is a n–dimensional Lebesgue measure on $\mathbf{R}^n$. Example can be found in [5].

References

[1] O.N. Capri and N.A. Fava, *"Strong Differentiability w.r.t. Product Measures"*, Studia Math. T.LXXVII (1984) p.173.

[2] R. Coifman and C. Fefferman, *"Weighted Norm Inequalities for Maximal Functions and Singular Integrals"*, Studia Math. 51 (1974) p.241.
[3] N. Fava, *"Weak Type Inequalities for Product Operators"*, Studia Math. T. XLII (1972) p.271.
[4] R. Fefferman, *"Strong Differentiation w.r.t. Measures"*, Am. J. of Math. vol.103 (1981) No.1, p.33.
[5] M. de Guzman, *"Differentiation of Integrals in* $\mathbf{R}^n$*"*, Lecture Note in Math. 481, Springer–Verlag 1975.
[6] B. Jessen, J. Marcinkiewicz, and A. Zygmund, *"Note on the Defferentiability of Multiple Integrals"*, Fund. Math. 25 (1935) p.217.

Department of Mathematics
Ewha Womans University
Seoul 132, Korea