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ON MINIMAL E–PROJECTION OF OPERATOR SPACES *

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Introduction

For a general unital C^*–algebra, the existence and uniqueness of its injective envelope was proved by Hamana [5]. Ruan [8] showed that an operator space E always has a unique injective envelope. The purpose of this paper is to investigate relations among E–seminorms, E–projections and its injective envelope.

1. Preliminaries

We let $M_{n\cdot m}$ be the space of complex $n \times m$ matrices and $M_n = M_{n\cdot n}$ for $n, m \in N$. $M_{n\cdot m}$ has the canonical basis $\{E_{ij}\}$ where E_{ij} is the matrix in $M_{n\cdot m}$ with 1 at (i,j) position and zero elsewhere.

Given a complex vector space E, we denote $M_{n\cdot m}(E) = E \otimes M_{n\cdot m}$ and $M_n(E) = E \otimes M_n$, the vector space of $n \times m$ and $n \times n$ matrices with entries in E.

For $x = [x_{ij}] \in M_n(E)$, $y = [y_{ij}] \in M_m(E)$, $\alpha = [\alpha_{ij}] \in M_{m\cdot n}$ and $\beta = [\beta_{ij}] \in M_{n\cdot m}$, we write

$$x \oplus y = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \in M_{n+m}(E)$$

and

$$\alpha x = \left[\sum_{j=1}^{n} \alpha_{ij} x_{jk}\right] \in M_{m\cdot n}(E), \ x\beta = [r_{ik}] \in M_{n\cdot m}(E)$$

where $r_{ik} = \sum_{j=1}^{n} \beta_{jk} x_{ij}$.

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For a linear map $\phi : E \to F$, we define the linear map $\phi_n : M_n(E) \to M_n(F)$ by

$$\phi_n([x_{ij}]) = [\phi(x_{ij})]$$

for $[x_{ij}] \in M_n(E)$.

A space with a matricial seminorm is a complex vector space E with a seminorm p_n on each $M_n(E)$ denoted as $(E,\ \{p_n\})$. If the seminorm p_n is actually a norm for each $n \in N$, we denote $(E\ \{p_n\})$ as $(E, \{\|\ \|_n\})$ and call space with a matricial norm.

Let $(E, \{\|\ \|_n\})$ and $(F, \{\|\ \|_n\})$ be two spaces with matricial norms and $\phi : E \to F$ a linear map. We write $\|\phi\|_{cb} = \sup\{\|\phi_n\| \text{ in } \in N\}$ and call ϕ a completely bounded map if $\|\phi\|_{cb} < \infty$, a complete contraction if $\|\phi\|_{cb} \leq 1$ and a complete isometry if each ϕ_n is an isometry.

2. Matricially Seminormed Spaces

Given a complex vector space E there is a natural way that each $M_n(E)$ is a M_n–bimodule with the module operations defined by $(\alpha, x) \to \alpha x$, $(x, \alpha) \to x\alpha$ for $\alpha \in M_n$, $x \in M_n(E)$. Hence for each linear map $\phi : E \to F$, $\phi_n : M_n(E) \to M_n(F)$ is an M_n–bimodule map, i.e., $\phi_n(\alpha x) = \alpha\phi_n(x)$ and $\phi_n(x\alpha) = \phi_n(x)\alpha$ for $\alpha \in M_n$, $x \in M_n(E)$. Regarding as $M_n = B(\mathbf{C}^n)$ and assuming the operator norm $\|\ \|$ on M_n, we have the following definition.

DEFNITION 2.1. *A m.s.n. space (matricially seminormed space) is a space with a matricial seminorm $(E,\ \{p_n\})$ which satisfies :*

(1) $p_{n+m}(x \oplus 0) = p_n(x)$ *for all* $x \in M_n(E)$, *and the zero element* $0 \in M_m(E)$ *and*

(2) $p_n(\alpha x) \leq \|\alpha\| p_n(x)$, $p_n(x\alpha) \leq \|\alpha\| p_n(x)$ *for all* $x \in M_n(E)$, $\alpha \in M_n$. *A m.s.n. space is L^∞ if it satisfies :*

(L^∞) $p_{n+m}(x \oplus y) = \max\{p_n(x), p_m(y)\}$

A. m.s.n. space is L^p $(1 \leq p < \infty)$ if it satisfies :

(L^p) $p_{n+m}(x \oplus y) = (p_n(x)^p + p_m(y)^p)^{\frac{1}{p}}$.

If the seminorm p_n is actually a norm for each $n \in N$, we denote $(E,\ \{p_n\})$ as $(E,\ \{\|\ \|_n\})$ and call a m.n.space (matricially normed space). If $(E, \{\| \ \|_n\})$ satisfies L^p–condition, we call $(E, \{\| \ \|_n\})$ L^p–m.n.space ($1 \leq p \leq \infty$).

Let $(E, \{\| \ \|_n\})$ and $(E, \{\| \ \|_n\})$ be two m.n.space and $\phi : E \to F$ a complete contraction. For each $n \in N$ we define seminorms P_n^ϕ on $M_n(E)$ by

$$p_n^\phi(x) = \|\phi_n(x)\|_n$$

for $x \in M_n(E)$. In particular, if $E = F$, we define seminorms q_n^ϕ on $M_n(E)$ by

$$q_n^\phi(x) = \lim_{k\to\infty} \sup \|(\frac{\phi_n + \cdots + \phi_n{}^k}{k})(x)\|_n.$$

for $x \in M_n(E)$. Then $(E, \{p_n^\phi\})$ and $(E, \{q_n^\phi\})$ are m.s.n.spaces with $q_n^\phi(x) \leq P_n^\phi(x) \leq \|x\|$ for all $x \in M_n(E)$. If $(F, \{\| \ \|_n\})$ is an L^∞–m.n.space, then $(E, \{p_n^\phi\})$ and $(E, \{q_n^\phi\})$ are L^∞–m.s.n.spaces ([8], Example 1.4.3. and 1.4.4.).

THEOREM 2.2 ([8] THEOREM 1.4.5.). *Let $(E, \{p_n\})$ be an L^∞–m.s.n. space, $(F, \{p_n\})$ a m.s.n.subspace of E and $\phi : F \to B(H)$ a linear map such that $\|\phi_n(x)\|_n \leq p_n(x)$ for all $x \in M_n(F)$ and $n \in N$. Then there exists a linear map $\overline{\phi} : E \to B(H)$ which extends ϕ and $\|\overline{\phi}_n(x)\|_n \leq p_n(x)$ for all $x \in M_n(E)$.*

Let H be a Hilbert space and $B(H)$ the von Neumann algebra of all bounded linear operators on H. We have an operator norm on $B(H)$ defined by

$$\|x\| = \sup\{\|x\eta\| : \|\eta\| \leq 1, \ \eta \in H\}.$$

Identifying $M_n(B(H))$ with $B(H^n)$, we get a operator norm on each $M_n(B(H))$ ($n \in N$). This family of norms $\{\| \ \|_n\}$ is called an operator matricial norm on $B(H)$. Obviously $(B(H), \{\| \ \|_n\})$ is an L^∞–m.s.space.

A linear subspace E of $B(H)$ with the above operator matricial norm is called an operator space. In particular, if $E = E^*$ and $I \in E$, we call E an operator system.

Let E be an operator system, A be a C^*–algebra and $\phi : E \to A$ be a linear map. Then we call ϕ is n–positive if ϕ_n is positive and we call ϕ completely positive if ϕ is n–positive for all n.

DEFINITION 2.3. *Let $B(H)$ be the von Neumann algebra of all bounded operators on a Hilbert sapce H. A linear map $\phi : B(H) \to B(H)$ is called a completely contractive projection if $\|\phi\|_{cb} \leq 1$ and $\phi^2 = \phi$. Let $(E, \{\| \ \|_n\})$ be an operator space contained in $B(H)$.*

An E–projection of $B(H)$ is a completely contractive projection $\phi : B(H) \to B(H)$ such that $\phi(x) = x$ for all $x \in E$.

Let P_H^E be the set of all E–projections of $B(H)$. Then P_H^E is non–empty. Define a partial ordering on P_H^E by saying

$$\psi \leq \phi \quad \text{if and only if} \quad \psi \circ \phi = \phi \circ \psi = \psi.$$

A minimal E–projection of $B(H)$ is an E–projection of $B(H)$ which is minimal under this partial ordering.

DEFINITION 2.4. *Let $(E, \{\| \ \|_n\})$ be an L^∞–m.n.space and $(F, \{\| \ \|_n\})$ a m.m.subspace of E. An L^∞–matricial seminorm $\{p_n\}$ on E is called an L^∞–matricial F–seminorm if it satisfies :*

(1) *$p_n(x) \leq \|x\|_n$ for all $x \in M_n(E)$*
(2) *$p_n(x) = \|x\|$ for all $x \in M_n(F)$.*

Let Γ_E^F be the set of all L^∞–matricial F–seminorms on E.
Then Γ_E^F is non–empty. Define a partial ordering on Γ_E^F by saying

$$\{p_n\} \leq \{q_n\} \text{ if and only if } p_n(x) \leq q_n(x)$$

for all $x \in M_n(E)$ and $n \in N$.

Let $(E, \{\| \ \|_n\})$ be an operator space contained in $B(H)$ and $\phi : B(H) \to B(H)$ a complete contraction such that $\phi(x) = x$ for all $x \in 2E$. Then the matricial seminorms $\{p_n^\phi\}$ and $\{q_n^\phi\}$ are L^∞–matricial seminorms on $B(H)$ and we have $\{q_n^\phi\} \leq \{p_n^\phi\}$.

LEMMA 2.5. *If $\psi \leq \phi$, then $\{p_n^\psi\} \leq \{p_n^\phi\}$.*

Proof. Given $x \in M_n(B(H))$, we have

$$p_n^\psi(x) = \|\psi_n(x)\|_n = \|\psi_n \circ \phi_n(x)\|_n \leq \|\phi_n(x)\|_n = p_n^\phi(x).$$

THEOREM 2.6. *Let $\phi \in P_H^E$ and $\{p_n^\phi\}$ be minimal in $\Gamma_{B(H)}^E$. Then ϕ is minimal in P_H^E.*

Proof. Let ψ be an arbitrary E–projection of $B(H)$ with $\psi \leq \phi$. By Lemma 2.5, $\{p_n^\psi\} \leq \{p_n^\phi\}$. Since $\{p_n^\phi\}$ is minimal, $\{p_n^\psi\} = \{p_n^\phi\}$. For all $x \in B(H)$, we have

$$\begin{aligned}\|\phi(x) - \psi(x)\}\|_1 &= \|\phi(x) - \phi \cdot \psi(x)\|_1 \\ &= \|\phi(x - \psi(x)\|_1 \\ &= p_1^\phi(x - \psi(x)) \\ &= p_1^\psi(x - \psi(x)) \\ &= 0.\end{aligned}$$

Hence $\psi = \phi$ and ϕ is minimal.

THEOREM 2.7. *Let $\{p_n^\psi\} \leq \{p_n^\phi\}$ and ϕ be minimal then $\{p_n^\psi\} = \{p_n^\psi\}$.*

Proof. Since ϕ is minimal $\phi \circ \psi \circ \phi = \phi$. Since $p_1^\psi(x - \phi(x)) = \|\psi(x - \phi(x))\| \leq p_1^\phi(x - \phi(x)) = 0$, $\psi = \psi \circ \phi$. Hence $\phi = \phi \circ \psi \circ \phi = \phi \circ \psi$. Therefore $\|\phi_n(x)\|_n = \|\phi_n \circ \psi_n(x)\|_n \leq \|\psi_n(x)\|_n$. This implies $\{p_n^\phi\} \leq \{p_n^\psi\}$ and $\{p_n^\phi\} = \{p_n^\psi\}$.

THEOREM 2.8 ([8] PROPOSITION 1.4.8.). *The partially ordered set Γ_F^E must have at least one minimal element.*

THEOREM 2.9. *Let E be a subspace of $B(H)$ and $\{p_n\}$ be a minimal element in $\Gamma_{B(H)}^E$. Then there exists a E–projection ϕ of $B(H)$ such that $p_n = p_n^\phi$ for each $n \in N$. In particular ϕ is minimal.*

Proof. Let $id : E \to B(H)$ be the inclusion map. Then $\|id(x)\|_n = \|x\|_n = p_n(x)$ for all $x \in M_n(E)$ and $n \in N$. By Theorem 2.2, there exists a linear map $\phi : B(H) \to B(H)$ which extends id and

$$\|\phi_n(x)\|_n \leq p_n(x) \leq \|x\|_n$$

for all $x \in M_n(B(H))$ and $n \in N$. Then $\{p_n^\phi\}$ and $\{q_n^\phi\}$ are L^∞–matricial E–seminorms on $B(H)$ such that

$$\{q_n^\phi\} \leq \{p_n^\phi\} \leq \{p_n\}.$$

Since $\{p_n\}$ is minimal, $\{p_n^\phi\} = \{p_n\} = \{q_n^\phi\}$. Hence

$$\begin{aligned}\|\phi(x) - \phi^2(x)\|_1 &= \|\phi(x - \phi(x)\| \\ &= p_1^\phi(x - \phi(x)) \\ &= q_1^\phi(x - \phi(x)) = 0.\end{aligned}$$

Therefore ϕ is an E–projection of $B(H)$. By Theorem 2.6, ϕ is minimal.

THEOREM 2.10. *Let ϕ be minimal. Then $\{p_n^\phi\}$ is minimal.*

Proof. By Theorem 2.8, there exists a minimal E–seminorm $\{p_n\}$ such that $\{p_n\} \leq \{p_n^\phi\}$. By Theorem 2.9, there exists a minimal E–projection ψ such that $\{p_n\} = \{p_n^\psi\}$. Hence $\{p_n^\psi\} \leq \{p_n^\phi\}$. Since ϕ is minimal, $\{p_n^\psi\} = \{p_n^\phi\}$ by Theorem 2.7. Therefore $\{p_n^\phi\}$ is minimal.

3. Injective operator spaces.

A C^*–algebra A is called injective if for any two C^*–algebras B and C and $B \subset C$ and any completely positive contraction $\phi : B \to A$ there is a completely positive contraction $\overline{\phi} : C \to A$ extending ϕ.

Let $E \subset B(H)$ be an operator space. E is called to be an injective operator space if for every operator space F, every operator subspace F_0 of F and every completely bounded map $\phi : F_0 \to E$, there exists a linear map $\overline{\phi} : F \to E$ such that $\overline{\phi}|_{F_0} = \phi$ and $\|\overline{\phi}\|_{cb} = \|\phi\|_{cb}$.

Let $E \subset B(H)$ be an operator system, E is called to be an injective operator system if for every operator system F, every operator subsystem F_0 of F and every completely positive contraction $\phi : F_0 \to E$, there exists a completely positive contraction $\overline{\phi} : F \to E$ with $\overline{\phi}|F_0 = \phi$.

It is well known that $B(H)$ is an injective operator space for arbitrary Hilbert space H ([8] Theorem 2.12.). Also $B(H)$ is an injective operator system ([7] Theorem 6.5.). Hence an operator space $E \subset B(H)$ is injective if and only if there exists an E–projection whose range is E and an operator system $E \subset B(H)$ is injective if and only if there exists a completely positive contraction $\phi : B(H) \to E$ such that $\phi|_E = id_E$. This implies that injective operator spaces (systems) must be norm closed.

Obviously, an injective operator system is injective operator space, and every injective C^*–algebra is an injective operator space. Furthermore, if A is an injective C^*–algebra and $p.q$ are projections in A, then pAq is an injective operator space.

THEOREM 3.1. *Let $E \subset B(H)$ be an operator space. Then E is injective if and only if there is a minimal E–projection such that $\phi(B(H)) = E$.*

Proof. ($\Longrightarrow$) Consider the identity map $i : E \to E$. Since E is injective, there exists a completely contractive map $\phi : B(H) \to E$ extending i. Obviously, ϕ is an E–projection and $\phi(B(H)) = E$. To complete the proof, we must show that ϕ is minimal. Let $\psi \leq \phi$, for each $x \in B(H)$, $\psi(x) = \phi(\psi(x)) \in E$ and $\phi(x) \in E$. Hence $\psi(x) - \phi(x) \in E$. Therefore $\psi(\psi(x) - \phi(x)) = \psi(x) - \phi(x)$. The other hand $\psi(\psi(x) - \phi(x)) = \psi(x) - \psi(\phi(x)) = 0$. Hence $\psi = \phi$.
($\Longleftarrow$) trivial.

COROLLARY 3.2. *Let $E \subset B(H)$ be injective and ϕ be an E–projection such that $\phi(B(H)) = E$. Then ϕ is minimal.*

COROLLARY 3.3. *Let ϕ be a projection. Put $\phi(B(H)) = E$. Then E is injective and ϕ is a minimal E–projection.*

COROLLARY 3.4. *If $\{E_\lambda\}_{\lambda \in \Lambda}$ is a family of operator spaces (systems), then $\oplus E_\lambda$ is injective if and only if each E_λ is injective.*

THEOREM 3.5 ([7] PROPOSITION 3.4.). *Let A and B be C^*–algebras with unit, let M be a subspace of A, $I \in M$, and let $S = M + M^*$. If $\phi : M \to B$ is unital and completely contractive, then $\overline{\phi} : S \to B$ given by $\overline{\phi}(a + b^*) = \phi(a) + \phi(b)^*$ is completely positive and completely contractive.*

THEOREM 3.6. *Let $I \in E \subset B(H)$ be an injective operator space. Then $E = E^*$ and E is an injective operator system.*

Proof. Since $I \in E$ is injective, there exists a minimal E–projection ϕ such that $\phi(B(H)) = E$ and ϕ is unital. By Theorem 3.5., ϕ is a completely positive E–projection. Hence $E = E^*$ and E is an injective operator system.

COROLLARY 3.7. *Let $E \subset B(H)$ be an operator system. If E is an injective operator space, E is an injective operator system.*

EXAMPLE 3.8. *Let $E = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbf{C} \right\}$. Then by Theorem 3.6, E is not injective and only $id; M_2 \to M_2$ is the E-projection. Hence the uniqueness of E-projection does not imply that E is injective.*

EXAMPLE 3.9. *Let $E = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbf{C} \right\}$. Then E is injective. But $E + E^* = \left\{ \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \mid a, b, c, \in \mathbf{C} \right\}$ is not injective since $\left\{ \begin{pmatrix} b & a \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbf{C} \right\}$ is not injective and $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & a \\ 0 & c \end{pmatrix}$.*

EXAMPLE 3.10. *Let $E = \left\{ \begin{pmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbf{C} \right\}$. Then E is injective. But $E + E^* + \mathbf{C}I = \left\{ \begin{pmatrix} a & b & c \\ d & f & 0 \\ e & 0 & f \end{pmatrix} \mid a, b, c, d, e, f \in \mathbf{C} \right\}$ is not injective, since*

$$\left\{ \begin{pmatrix} b & a & c \\ f & d & 0 \\ 0 & e & f \end{pmatrix} \mid a, b, c, d, e, f \in \mathbf{C} \right\} \quad \text{is not injective}$$

$$\text{and} \quad \begin{pmatrix} a & b & c \\ d & f & 0 \\ e & 0 & f \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} b & a & c \\ f & d & 0 \\ 0 & e & f \end{pmatrix}.$$

EXAMPLE 3.11. *Let $H = \mathbf{C}^2$, $B(H) = M_2$ and $E = \{aI \mid a \in \mathbf{C}\} \subset M_2$. Define $\phi_k : M_2 \to E$ as $\phi_k(a_{ij}) = a_{kk}I$ $(k = 1, 2)$. Then ϕ_1 and ϕ_2 are completely positive unital contraction, $\phi_1(M_2) = \phi_2(M_2) = E$ and E is injective. Hence ϕ_1 and ϕ_2 are minimal E-projections. Therefore injective operator space does not imply the uniqueness of E-projection.*

Let E be an operator space. Then E is contained in some injective operator space $B(H)$.

DEFINITION 3.12. *Let E be an operator space. An extension of E is a pair (Z, k) of an operator space Z and a completely isometric embedding $k : E \to Z$. An extension (Z, k) of E is called injective if Z is an injective operator space. An extension (Z, k) of E is called an injective envelope of E if (Z, k) is an injective extension of E and id is the only complete contraction (from Z into Z) which extends $id_{k(E)} : k(E) \to Z$ from $k(E)$ to Z.*

THEOREM 3.13. *Let $E \subset B(H)$ be an operator space and ϕ be a minimal E-projection. Then $\phi(B(H))$ is an injective enbelope of E.*

Proof. The same as the proof of [8] Theorem 2.2.2.

THEOREM 3.14 ([8] PROPOSITION 2.2.5.). *Let E be an operator space, $E \subset Z \subset B(H)$ an injective envelope of E. Then there exists a minimal E-projection ϕ of $B(H)$ such that $\phi(B(H)) = Z$.*

THEOREM 3.15 ([8] THEOREM 2.2.6). *Let E be a operator space, $E \subset Z$ be an injective. Then Z is an injective envelope of E if and only if injective subspace of Z containing E is Z itself.*

THEOREM 3.16. *Let E be an operator space, ϕ be a minimal E-projection and ψ be a E-projection such that $\phi(B(H)) = \psi(B(H))$. Then ψ is minimal.*

Proof. Let ρ be a minimal E-projection of $B(H)$ with $\rho \leq \psi$. Then $E \subset \rho(B(H)) \subset \psi(B(H)) = \phi(B(H))$. By Theorem 3.13, $\rho(B(H))$ is an injective envelope of E. Hence By Theorem 3.15, $\rho(B(H)) = \phi(B(H)) = \psi(B(H))$. Therefore $\rho = \psi$.

THEOREM 3.17. *Let $E \subset B(H)$ be an operator space and ϕ be a E-projection. Then ϕ is minimal if and only if $\phi(B(H))$ is an injective envelope of E.*

Proof. ($\Longrightarrow$) Theorem 3.13.

($\Longleftarrow$) By Theorem 3.14, there is a minimal E–projection ψ of $B(H)$ such that $\psi(B(H)) = \phi(B(H))$. Hence by Theorem 3.16, ϕ is minimal.

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