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ON A VECTOR FIELD PROBLEM OVER A LENS SPACE *

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For a real vector bundle ξ, we denote by *span* ξ the maximum number of linearly independent cross sections of ξ. In particular, we denote span M = span τM, where τM is the tangent bundle of a smooth manifold M. Let $\oplus$ denote Whitney sum of vector bundles. According to context, the integer m will denote either itself or the trivial m-plane bundle over an appropriate space. The *stable span* of a smooth manifold M is defined by

$$\text{st.span } M = \text{span}(\tau M \oplus 1) - 1 = \text{span}(M \times P) - p$$

for any p-dimensional parallelizable manifold P. Note that st.span M^n $= n$ if and only if M^n is stably parallelizable, and span $M^n = n$ if and only if M^n is parallelizable.

Let p be an odd prime and n a non-negative integer, and let $L^{2n+1}(p; a_1, a_2, \ldots, a_{n+1})$ denote a generalized lens space. If all the a_i are equal to 1, then $L^{2n+1}(p; a_1, a_2, \ldots, a_{n+1})$ is known as a standard lens space, denoted by $L^{2n+1}(p)$ simply. Associated with the principal Z_p bundle $\pi : S^{2n+1} \to L^{2n+1}(p; a_1, a_2, \ldots, a_{n+1})$ one may form a complex line bundle γ over $L^{2n+1}(p; a_1, a_2, \ldots, a_{n+1})$ by dividing out the diagonal actions of Z_p on $S^{2n+1} \times \mathbf{C}$, where the generator of Z_p acts on $\mathbf{C}$ by multiplication by $\exp(2\pi i/p)$. There are also the similarly defined line bundles where Z_p acts on $\mathbf{C}$ by multiplication by $\exp(2\pi bi/p)$ which are just the complex tensor power γ^b. The tangent bundle of a lens space is described as follows :

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THEOREM 1 ([7]). *$\tau L^{2n+1}(p; a_1, a_2, \ldots, a_{n+1}) \oplus 1$ is isomorphic to $re(\gamma^{a_1} \oplus \gamma^{a_2} \oplus \cdots \oplus \gamma^{a_{n+1}})$ over $L^{2n+1}(p; a_1, a_2, \ldots, a_{n+1})$, where re denotes the realification of a complex vector bundle.*

For the canonical complex line bundle γ (will be denoted by $\gamma_{n,p}$ from now on) over the standard lens space $L^{2n+1}(p)$, to determine the span of $\mathrm{re}(m\gamma_{n,p}) = \mathrm{re}(\gamma_{n,p} \oplus \cdots \oplus \gamma_{n,p})$, ($m$ times Whitney sum) is the generalized vector field problem. Let $\eta_{n,p}$ denote the realification of $\gamma_{n,p}$, so that $m\eta_{n,p} = \mathrm{re}(m\gamma_{n,p})$ for any m. In this paper we will give a characterization of a lower bound of the span of $m\eta_{n,p}$ and some estimation of the span of $m\eta_{n,p}$.

Let $X = (X, T_p)$ be a Hausdorff space with a free cyclic group action of order p generated by an action T_p. The *index* of (X, T_p) is the largest integer $2n+1$ for which there is an equivariant map of the $(2n+1)$–sphere S^{2n+1} into X. The *coindex* of (X, T_p) is the least integer $2n+1$ for which there is an equivariant map of X into S^{2n+1}. Here S^{2n+1} is assumed to have the standard linear Z_p action. Note that an even dimensional sphere does not admit a free Z_p action for an odd prime p, because if there would be a free Z_p action on even dimension sphere, the Euler characteristic 2 of the sphere would be p times the Euler characteristic of its quotient space, which is impossible.

Let $V_k(\mathbf{C}^n)$ be the (complex) Stiefel manifold of orthonormal k–frames in the n–dimensional complex space $\mathbf{C}^n$. We define a free Z_p action T_p on $V_k(\mathbf{C}^n)$ such that $\exp(2\pi i/p)$ acts on each vector of k–frames. Similarly, we can define a free Z_p action on the real Stiefel manifold $V_k(\mathbf{R}^{2n})$.

Let $v_1, v_2, \ldots, v_k$ be linearly independent real vectors in $\mathbf{R}^{2n}$ and let $GS(v_1, v_2, \ldots, v_k) = (v_1', v_2', \ldots, v_k')$ be the result of the Gram–Schmidt Orthogonalization process, i.e.,

$$v_1' = \frac{v_1}{\|v_1\|},$$

and

$$v_j' = \frac{v_j - \sum_{i<j} <v_i, v_i'> v_i'}{\|v_j - \sum_{i<j} <v_i, v_i'> v_i'\|}, \quad j \geq 2.$$

Then, GS is an equivariant map on $V_k(\mathbf{R}^{2n})$. Indeed, if we let $GS(e^{i\theta}v_1, e^{i\theta}v_2, \ldots, e^{i\theta}v_k) = (w_1, w_2, \ldots, w_k)$, then

$$w_1 = \frac{e^{i\theta}v_1}{\|e^{i\theta}v_1\|} = e^{i\theta}\frac{v_1}{\|v_1\|} = e^{i\theta}v_1',$$

and

$$\begin{aligned} w_j &= \frac{e^{i\theta}v_j - \sum_{i<j} < e^{i\theta}v_i, e^{i\theta}v_i' > e^{i\theta}v_i'}{\|e^{i\theta}v_j - \sum_{i<j} < e^{i\theta}v_i, e^{i\theta}v_i' > e^{i\theta}v_i'\|} \\ &= e^{i\theta}\frac{v_j - \sum_{i<j} < v_i, v_i' > v_i'}{\|v_j - \sum_{i<j} < v_i, v_i' > v_i'\|} \\ &= e^{i\theta}v_j' \quad \text{for} \quad j \geq 2. \end{aligned}$$

The problem to estimate the lower bound of span$(m\gamma_{n,p})$ can be characterized as follows :

THEOREM 2. *The following statements are equivalent:*

(1) *span*$(m\gamma_{n,p}) \geq k$,
(2) *there is a* Z_p *equivariant map from* S^{2n+1} *into* $V_k(\mathbf{C}^m)$, *i.e., index*$(V_k(\mathbf{C}^m), T_p) \geq 2n+1$,
(3) *there is a map* $\phi : \mathbf{C}^{n+1} \times \mathbf{C}^k \to \mathbf{C}^m$ *such that*

i) $\phi(e^{i\theta}z, w) = \phi(z, e^{i\theta}w) = e^{i\theta}\phi(z, w)$ *for each* $e^{i\theta}$ *in* Z_p,
ii) $\phi(z, w) = 0$ *iff* $z = 0$ *or* $w = 0$,
iii) $\phi(z, aw_1 + bw_2) = a\phi(z, w_1) + b\phi(z, w_2)$ *for any complex numbers* a *and* b.

Proof. (1) $\Leftrightarrow$ (2). Let $\tilde{T}$ be the Z_p action on $S^{2n+1} \times V_k(\mathbf{C}^m)$ defined by $\tilde{T}(z, v) = (T(z), T_p(v))$ for $z \in S^{2n+1}$ and $v \in V_k(\mathbf{C}^m)$, and let π be the map from $S^{2n+1} \times V_k(\mathbf{C}^m)/\tilde{T}$ onto $L^{2n+1}(p) = S^{2n+1}/T$ induced by the projection from $S^{2n+1} \times V_k(\mathbf{C}^m)$ onto S^{2n+1}. Then π is the projection of the k-frame bundle associated with $m\gamma_{n,p}$. And the existence of a cross section of this bundle is equivalent to the existence of an equivariant map from S^{2n+1} to $V_k(\mathbf{C}^m)$.

(2) $\Rightarrow$ (3). Let f be a Z_p equivariant map from S^{2n+1} to $V_k(\mathbf{C}^m)$, and we define a map $\phi : \mathbf{C}^{n+1} \times \mathbf{C}^k \to \mathbf{C}^m$ by

$$\phi(z,w) = \begin{cases} f(\frac{z}{\|z\|}) \bullet \|z\|w & \text{if } z \neq 0 \\ 0 & \text{if } z = 0, \end{cases}$$

where $\bullet$ means the matrix multiplication and $f(\frac{z}{\|z\|}) \in V_k(\mathbf{C}^m)$ is regarded as an $m \times k$ complex matrix, and w is a column vector in $\mathbf{C}^k$. Then ϕ is clearly continuous and satisfies (i), (ii), and (iii). Indeed, $\phi(e^{i\theta}z, w) = f(e^{i\theta}\frac{z}{\|z\|}) \bullet \|z\|w = e^{i\theta}f(\frac{z}{\|z\|}) \bullet \|z\|w = e^{i\theta}\phi(z,w) = f(\frac{z}{\|z\|}) \bullet e^{i\theta}\|z\|w = \phi(z, e^{i\theta}w)$ for any $e^{i\theta}$. If $z \neq 0$ and $\phi(z,w) = f(\frac{z}{\|z\|}) \bullet \|z\|w = 0$, then $w = 0$, because $f(\frac{z}{\|z\|})$ has rank k. Clearly $\phi(z, aw_1 + bw_2) = a\phi(z,w_1) + b\phi(z,w_2)$ for any complex a and b.

(3) $\Rightarrow$ (2). Let $\phi : \mathbf{C}^{n+1} \times \mathbf{C}^k \to \mathbf{C}^m$ be the map given in (3) and let $z \in S^{2n+1}$ be any element. Then $\phi(z,e_1), \phi(z,e_2), \ldots, \phi(z,e_k)$ are linearly independent, where $e_1, e_2, \ldots, e_k$ denotes the standard basis for $\mathbf{C}^k$. Then $g(z) = GS(\phi(z,e_1), \phi(z,e_2), \ldots, \phi(z,e_k))$ is the desired Z_p equivariant map from S^{2n+1} to $V_k(\mathbf{C}^m)$.

Similarly, we can prove the following theorem:

THEOREM 3. *The following statements are equivalent:*

(1) span$(m\eta_{n,p}) \geq k$,

(2) there is a Z_p equivariant map from S^{2n+1} into $V_k(\mathbf{R}^{2m})$, i.e., index $(V_k(\mathbf{R}^{2m}), T_p) \geq 2n+1$,

(3) there is a map $\phi : \mathbf{R}^{2n+2} \times \mathbf{R}^k \to \mathbf{R}^{2m}$ such that

i) $\phi(e^{i\theta}x, y) = e^{i\theta}\phi(x,y)$ *for each* $e^{i\theta} \in Z_p$, $x \in \mathbf{R}^{2n+1}$ *and* $y \in \mathbf{R}^k$,

ii) $\phi(x,y) = 0$ *iff* $x = 0$ *or* $y = 0$

iii) $\phi(x, ay_1 + by_2) = a\phi(x,y_1) + b\phi(x,y_2)$ *for any real numbers a and b.*

Since $(n+1)\eta_{n,p} = \tau L^{2n+1}(p) \oplus 1$ has a nontrivial cross section, span$(m\eta_{n,p}) \geq 1$ for any $m \geq n+1$. For $m < n+1$, if span$(m\eta_{n,p}) \geq 1$, then by Theorem 3, there exists a Z_p equivariant map f from S^{2n+1} to $V_1(\mathbf{R}^{2m}) = S^{2m-1}$. But there is no map from S^k to S^l which commutes with some free actions of a nontrivial finite group on the spheres S^k, S^l

if $k > l$, by a generalized theorem of Borsuk and Ulam (for example, see [2]). Hence we have

THEOREM 4. *span*$(m\eta_{n,p}) = 0$ *if and only if* $n \geq m$.

Therefore, we can restrict our concern to find span$(m\eta_{n,p})$ for $m > n$ from now on.

THEOREM 5. *If a* $2k$*–connected space* X *admits a free* Z_p *action, then there exists a* Z_p *equivariant map from* S^{2k+1} *to* X.

Proof. Let $k = 0$ to prove it by induction on k. With the identification

$$Z_p = \{\exp(2\pi in/p) : n = 0, 1, \ldots, p-1\}$$

as the subgroup of S^1, we define a map $f : Z_p \to X$ by

$$f(1) = x_0$$

and

$$f(\exp(2\pi in/p)) = \exp(2\pi in/p)x_0,$$

where x_0 is a fixed element in X. Since $\Pi_0(X) = 0$, f has an extension

$$\overline{f} : \{z \in S^1 : 0 \leq \arg z \leq 2\pi/p\} \to X.$$

Now, we define $\tilde{f} : S^1 \to X$ by

$$\tilde{f}(z) = \exp(2\pi ih/p)\overline{f}(\exp(-2\pi ih/p)z),$$

where $2\pi h/p \leq \arg z \leq 2\pi(h+1)/p, h = 1, 2, \ldots, p-1$. Then $\tilde{f}$ is clearly continuous and Z_p equivariant. Suppose that the theorem is true for $k < n$, and let X be $2n$–connected. First, we divide the sphere S^{2n+1} into p subsets:

$$S_b^{2n+1} = \{(z_1, \ldots, z_{n+1}) \in S^{2n+1} : 2\pi b/p \leq \arg z_{n+1} \leq 2\pi(b+1)/p\},$$
$$b = 0, 1, \ldots, p-1.$$

We can see that the boundary $\partial(S_0^{2n+1})$ of S_0^{2n+1} is the union of

$$B_1 = \{(z_1, \dots, z_{n+1}) \in S^{2n+1} : \arg z_{n+1} = 0\}$$

and

$$B_2 = \{(z_1, \dots, z_{n+1}) \in S^{2n+1} : \arg z_{n+1} = 2\pi/p\},$$

and each B_i is homeomorphic to the $2n$–dimensional standard ball. Hence, by the induction hypothesis, there exists a Z_p equivariant map f from the boundary $\partial(B_1)$ of B_1 to X and this f has an extension f_1 from B_1 to X, because X is $2n-1$ connected. And we extend f_1 to f_2 over $B_1 \cup B_2$ by defining

$$f_2(z) = \exp(2\pi i/p) f_1(\exp(-2\pi i/p)z)$$

for $z \in B_2$. This map f_2 is well–defined continuous map, because $B_1 \cap B_2 = S^{2n-1} \subset S^{2n+1}$ is the domain of f. Since $B_1 \cup B_2$ is homeomorphic to S^{2n} and X is $2n$–connected, f_2 has an extension $\overline{f}$ over S_0^{2n+1}. Now we extend $\overline{f}$ over S^{2n+1} to get a desired Z_p equivariant map $\tilde{f}$ as follows:

$$\tilde{f}(z) = \exp(2\pi ib/p)\overline{f}(\exp(-2\pi ib/p)z) \text{ for } z \in S_b^{2n+1}, \ b = 1, 2, \dots, p-1.$$

Actually, it is trivial to show that the map $\tilde{f}$ is Z_p equivariant from the construction of $\tilde{f}$.

COROLLARY 6. *span*$(m\eta_{n,p}) \geq 2(m-n)$ *for all* m.

Proof. Since $V_k(\mathbf{C}^n)$ is $2(n-k)$–connected, there exists a Z_p equivariant map f from S^{2n+1} to $V_{m-n}(\mathbf{C}^m)$.
Let $\tilde{f} : V_{m-n}(\mathbf{C}^m) \to V_{2(m-n)}(\mathbf{R}^{2m})$ be the map defined by

$$\tilde{f}(v_1, \dots, v_{m-n}) = (re(v_1), re(iv_1), \dots, re(v_{m-n}), re(iv_{m-n})),$$

where $re(z_1, z_2, \dots, z_m) = (x_1, y_1, x_2, y_2, \dots, x_m, y_m)$ and $z_j = x_j + iy_j$, $j = 1, 2, \dots, m$. Then $\tilde{f}$ is a Z_p equivariant map, and so does $\tilde{f} \circ f : S^{2n+1} \to V_{2(m-n)}(\mathbf{R}^{2m})$. Thus, we get span$(m\eta_{n,p}) \geq 2(m-n)$.

Let ζ be a real $2m$–dimensional vector bundle over the lens space $L^{2n+1}(p)$ with $m > n$, and let ζ have k linearly independent cross sections, then the geometric dimension of $\zeta - 2m$ is less than $2m - k$. Hence, the Atiyah's γ–operator satisfies

$$\begin{aligned}\gamma_t(\zeta - 2m) &= \sum_{i \geq 0} \gamma^i(\zeta - 2m)t^i \\ &= 1 + \gamma^1(\zeta - 2m)t + \gamma^2(\zeta - 2m)t^2 + \cdots,\end{aligned}$$

and

$$\gamma^i(\zeta - 2m) = 0 \quad \text{for} \quad i \geq 2m - k.$$

Now for $\zeta = m\eta_{n,p}$, we have (cf. [5])

$$\gamma_t(\overline{\eta}_{n,p}) = 1 + \overline{\eta}_{n,p}t - \overline{\eta}_{n,p}t^2,$$

where

$$\overline{\eta}_{n,p} = \eta_{n,p} - 2 \in \widetilde{KO}(L^{2n+1}(p)).$$

Hence,

$$\begin{aligned}\gamma_t(m\overline{\eta}_{n,p}) &= (1 + \overline{\eta}_{n,p}t - \overline{\eta}_{n,p}t^2)^m \\ &= \sum_{i=0}^{m} \binom{m}{i} (\overline{\eta}_{n,p})^i (t - t^2)^i\end{aligned}$$

in $\widetilde{KO}(L^{2n+1}(p))$, and the γ-dimension on $m\overline{\eta}_{n,p}$ is $\dim_\gamma(m\overline{\eta}_{n,p}) = 2\sup\{i | \binom{m}{i}(\overline{\eta}_{n,p})^i \neq 0\}$.

Kambe [5] computed $\widetilde{KO}(L^{2n+1}(p))$.

THEOREM 7. *Let* $q = \frac{1}{2}(p-1)$ *and* $n = s(p-1) + r$, $0 \leq r < p - 1$. *Then*

$$\widetilde{KO}(L^{2n+1}(p)) \cong \begin{cases} (Z_{p^{s+1}})^{[\frac{r}{2}]} \oplus (Z_{p^s})^{q-[\frac{r}{2}]} & \text{if } n \not\equiv 0 \pmod 4 \\ Z_2 \oplus (Z_{p^{s+1}})^{[\frac{r}{2}]} \oplus (Z_{p^s})^{q-[\frac{r}{2}]} & \text{if } n \equiv 0 \pmod 4, \end{cases}$$

and the direct summand $(Z_{p^{s+1}})^{[\frac{r}{2}]}$ *and* $(Z_{p^s})^{q-[\frac{r}{2}]}$ *are generated additively by* $\overline{\eta}_{n,p}, \ldots, (\overline{\eta}_{n,p})^{[\frac{r}{2}]}$ *and* $(\overline{\eta}_{n,p})^{[\frac{r}{2}]+1}, \ldots, (\overline{\eta}_{n,p})^q$ *respectively.*

Moreover its ring structure is given by

$$(\overline{\eta}_{n,p})^{q+1} = \sum_{i=0}^{q} \frac{-(2q+1)}{2i-1} \binom{q+i+1}{2i-2} (\overline{\eta}_{n,p})^i$$

and

$$(\overline{\eta}_{n,p})^{[\frac{n}{2}]+1} = 0.$$

From the group $\widetilde{KO}(L^{2n+1}(p))$, we can see that the order of the element $\overline{\eta}_{n,p}^i$ is $p^{1+[\frac{n-2i}{p-1}]}$ for $1 \le i \le [\frac{n}{2}]$, and then

$$\dim_\gamma(m\overline{\eta}_{n,p}) = 2\sup\{i | \binom{m}{i} \neq 0 \bmod (p^{1+[\frac{n-2i}{p-1}]})\}.$$

Therefore, we have

THEOREM 8.

$$\begin{aligned} span(m\eta_{n,p}) &\le 2m - \dim_\gamma(m\overline{\eta}_{n,p}) \\ &= 2\{m - \sup\{i | \binom{m}{i} \neq 0 \mod (p^{1+[\frac{n-2i}{p-1}]})\}\}. \end{aligned}$$

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