Comm. Korean Math. Soc. 4 (1989), No. 2, pp. 297~302

THE ROLE OF THE OUTER FUNCTIONS IN THE FUNCTIONAL CALCULUS *

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1. Introduction.

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$, let $\mathcal{A}_T$ denote the smallest subalgebra of $\mathcal{L}(\mathcal{H})$ that contains T and $1_{\mathcal{H}}$ and is closed in the ultraweak operator topology ($\mathcal{A}_T$ is called a dual algebra generated by T). Moreover, let $\mathcal{Q}_T$ denote the quotient space $(\tau c)/{}^{\perp}\mathcal{A}_T$, where (τc) is the trace–class ideal in $\mathcal{L}(\mathcal{H})$ under the trace norm, and ${}^{\perp}\mathcal{A}_T$ denotes the preannihilator of $\mathcal{A}_T$ in (τc).

One knows (cf. [3]) that $\mathcal{A}_T$ is the dual space of $\mathcal{Q}_T$ and that the duality is given by

$$<A,[L]>=\operatorname{tr}(AL), \quad A \in \mathcal{A}_T, \quad [L] \in \mathcal{Q}_T$$

where $[L]$ is the image in $\mathcal{Q}_T$ of the operator L in (τc).

If x and y are vectors in $\mathcal{H}$ and we write, as usual, $x \otimes y$ for the rank–one operator in (τc) defined by $(x \otimes y)(u) = (u, y)x$, $u \in \mathcal{H}$, then $[x \otimes y] \in \mathcal{Q}_T$ and an easy claculation shows that for any A in $\mathcal{A}_T$ we have

$$<A,[x \otimes y]>=\operatorname{tr}(A(x \otimes y)) = (Ax, y).$$

It is well known that every element of $\mathcal{Q}_T$ has the form $\sum_1^\infty [x_i \otimes y_i]$. A dual algebra $\mathcal{A}_T$ is said to have property (A_1) if every element $[L]$ of $\mathcal{Q}_T$ can be written in the form $[L] = [x \otimes y]$ for certain vectors x, y in $\mathcal{H}$.

Received Feb. 7, 1989. Revised April 30, 1989.
*이 논문은 1988년도 문교부지원 한국학술진흥재단의 자유공모과제 학술연구조성비에 의하여 연구되었음.

If T is a contraction in $\mathcal{L}(\mathcal{H})$, then T can be written as a direct sum $T = T_1 \oplus T_2$ where T_1 is a completely nonunitary contraction (i.e. T_1 has no nontrivial invariant subspace on which it acts as a unitary operator) and T_2 is a unitary operator. If T_2 is absolutely continuous or acts on the space (0), T will be called an absolutely continuous contraction. For absolutely continuous contraction T, one knows (cf. [2], Theorem 4.1) that there is a functional calculus $\Phi_T : H^\infty \to \mathcal{A}_T$ defined by $\Phi(u) = u(T)$ $(= \lim_{r\to 1} a_n r^n T^n)$ for every $u(z) = \sum_0^\infty a_n z^n \in H^\infty$. In this paper, using the property that a minimal unitary dilation of an absolutely continuous contraction is absolutely continuous, we study the image of Sz.–Nagy and Foias functional $u(T)$.

2. Preliminaries.

Let H^p $(0 < p \leq \infty)$ be the Hardy class of functions holomorpic on $U = \{z \mid |z| < 1\}$ such that the corresponding norm

$$\|f\|_p = \begin{cases} \sup\limits_{0\leq r<1} [\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta]^{1/p} & (0 < p < \infty) \\ \sup\limits_{z\in U} |f(z)| & (p = \infty) \end{cases}$$

is finite.

We call a function u defined on U an outer function if it admits a representation of the form

$$u(z) = c\exp\left[\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\log k(t)dt\right] \quad (z \in U)$$

where $k(t) \geq 0$, $\log k(t) \in L^1$, and c is a complex number of modulus 1.

The class of the outer functions belongs to H^p will be denoted by E^p. If $D = \{z \mid |z| = 1\}$, we call a function $u \in H^\infty$ an inner function if u satisfies the condition

$$|u(e^{it})| = 1 \quad \text{a.e. on } D.$$

Then it is well known that every function $u \in H^p$ $(0 < p \leq \infty)$ such that $u \not\equiv 0$ has a canonical factorization

$$u = u_i u_e$$

into the product of an inner function u_i and an outer function u_e, which are determined up to constant factor of modulus 1.

For two operators, $\mathcal{A}$ on the Hilbert space $\mathcal{H}_1$, and $\mathcal{B}$ on the Hilbert space $\mathcal{H}_2$, we shall denote by

$$\mathcal{A} = pr\mathcal{B}$$

the following relation:

(i) $\mathcal{H}_1$ is a subspace of $\mathcal{H}_2$, and

(ii) $\mathcal{A}a = P_{\mathcal{H}_1}\mathcal{B}a$ for all $a \in \mathcal{H}_1$, where $P_{\mathcal{H}_1}$ denotes the orthogonal projection from $\mathcal{H}_2$ into $\mathcal{H}_1$.

Then we call $\mathcal{B}$ a dilation of $\mathcal{A}$ if

$$\mathcal{A}^n = pr\mathcal{B}^n \quad \text{for } n = 1, 2, \ldots.$$

3. Some results of the functional calculus.

THEOREM 3.1. *Suppose T is an absolutely continuous contraction on $\mathcal{H}$. Then its minimal unitary dilation is absolutely continuous.*

Proof. It is well known that there exist unitary T_0 on $\mathcal{H}_0$ and completely nonunitary T_1 on $\mathcal{H}_1$ such that $T = T_0 \oplus T_1$ and $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, where

$$\mathcal{H}_0 = \{h \mid \|T^n h\| = \|h\| = \|T^{*n} h\|\}$$

Since T is absolutely continuous, T_0 is absolutely continuous. Let $U_1 \in \mathcal{L}(\mathcal{K}_1)$ be the minimal unitary dilation of T_1, where $\mathcal{K}_1 = \bigvee_{-\infty}^{\infty} U_1^n \mathcal{H}_1$, then U_1 is absolutely continuous. Let $U = T_0 \oplus U_1$ and $\mathcal{K} = \mathcal{H}_0 \oplus \mathcal{K}_1$. Then

$$\begin{aligned} U^*U &= (T_0 \oplus U_1)^*(T_0 \oplus U_1) \\ &= (T_0^* \oplus U_1^*)(T_0 \oplus U_1) \\ &= T_0^*T_0 \oplus U_1^*U_1 \\ &= 1_{\mathcal{H}_0} \oplus 1_{\mathcal{K}_1} \\ &= 1_{\mathcal{K}} = UU^*. \end{aligned}$$

Hence U is unitary. Since

$$\begin{aligned} P_{\mathcal{H}}U^n h &= P_{\mathcal{H}}(T_0^n \oplus U_1^n)(h_0 \oplus k) \\ &= P_{\mathcal{H}}(T_0^n h_0 \oplus U_1^n k) \\ &= T_0^n h_0 \oplus P_{\mathcal{H}_1} U_1^n k \\ &= T_0^n h_0 \oplus T_1^n k \\ &= (T_0^n \oplus T_1^n)(h_0 \oplus k) \\ &= T^n h, \quad \text{for } h = h_0 \oplus k,\ h_0 \in \mathcal{H}_0, \text{ and } k \in \mathcal{K}_1, \end{aligned}$$

U is a dilation of T. Also

$$(*) \qquad \begin{aligned} U^n\mathcal{H} &= U^n(\mathcal{H}_0 \oplus \mathcal{H}_1) \\ &= (T_0^n \oplus U_1^n)(\mathcal{H}_0 \oplus \mathcal{H}_1) \\ &= T_0^n\mathcal{H}_0 \oplus U_1^n\mathcal{H}_1. \end{aligned}$$

The equation (*) implies that $\mathcal{K} = \bigvee_{-\infty}^{\infty} U^n\mathcal{H}$.

Then $U \in \mathcal{L}(\mathcal{K})$ is a minimal unitary dilation and absolutely continuous, since T_0 and U_1 are absolutely continuous.

COROLLARY 3.2. *For every absolutely continuous contraction T on $\mathcal{H}$, and for every outer function $u \in H^\infty$ (i.e. $u \in E^\infty$), the operator $u(T)$ has an inverse with domain dense in $\mathcal{H}$.*

Proof. Proposition 3.1 ([5], p.118) and Theorem 3.1.

COROLLARY 3.3. *Suppose T is an absolutely continuous contraction on $\mathcal{H}$. Then $\Phi_T(E^\infty)$ is contained in the set of one–to–one operators on $\mathcal{H}$.*

THEOREM 3.4. *For every nonzero non–outer function $u \in H^\infty$ there exists a contraction T on a space $\mathcal{H} \neq (0)$ such that $\mathcal{A}_T$ has property (A_1) and $u(T) = 0$.*

Proof. Let $u = u_i u_e$ be the canonical factorization of u, where u_i is an inner function and u_e is an outer function. Then $u_i H^2$ is a subspace of H^2. Setting

$$\mathcal{H} = H^2 \cap (u_i H^2)^{\perp}$$

we have $\mathcal{H} \neq (0)$.
Let V denote multiplication of H^2 by the variable λ, then V is a unilateral shift on H^2, and consequently $V^{*n} \to 0$ as $n \to \infty$. Since $u_i H^2$ is invariant for V, its orthogonal complement in $\mathcal{H}$ is invariant for V^*. Setting

$$T = (V^* \mid_{\mathcal{H}})^*$$

we obtain thus a contraction on $\mathcal{H}$ such that

$$T^* = V^* \mid_{\mathcal{H}} = V^*_{\mathcal{H}}.$$

If dim $\mathcal{H} < \infty$, by [2, Theorem 2.06.], $\mathcal{A}_T$ has property (A_1). If dim $\mathcal{H} = \infty$, by [1, Theorem 3.2] and $A_1(\mathcal{H})$ is self-adjoint, $T^* = V^*_{\mathcal{H}} \in A_1(\mathcal{H})$. Hence $T \in A_1(\mathcal{H})$. (See [2] for $A_1(\mathcal{H})$.) Therefore $\mathcal{A}_T$ has property (A_1). Next since $T^n = PV^n|_{\mathcal{H}}$ $(n = 0, 1, 2, \dots)$, where P denotes the orthogonal projection of H^2 onto $\mathcal{H}$. Hence $u_i(T)h = Pu_i(V)h = Pu_i h$ for all $h \in \mathcal{H}$.
Since $u_i h \in u_i H^2$ and hence $u_i h \perp \mathcal{H}$ for every $h \in H^2$, we obtain that $u_i(T) = 0$. Therefore $u(T) = u_i(T)u_e(T) = 0$.

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