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SOME CONSEQUENCES OF CHERN'S KINEMATIC FORMULA IN NON-EUCLIDEAN SPACE

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1. Introduction

Let P be a compact orientable p-dimensional Riemannian manifold which is imbedded in n-dimensional non-Euclidean space $E^n(K)$ of constant curvature K(briefly $P \subset E^n(K)$). Denote by R_P and R_K (briefly K) the curvature tensor fields of P and $E^n(K)$ respectively.

Recently Ishihara [4] and Lee [5] obtained the following result which generalizes Chern's kinematic formula in Euclidean space [2]. Let $P \subset E^n(K)$ and $Q \subset E^n(K)$ be compact submanifolds of dimensions p and q respectively. Let $E_K(n)$ be the group of proper motions of $E^n(K)$ and dg the standard kinematic density on $E_K(n)$. If $0 \leq e$ even $\leq p+q-n$ and $g \in E_K(n)$, then

$$(1) \qquad \int \mu_e(P \cap gQ, K)\, dg = \sum_{0 \leq i \text{ even } \leq e} c_{e,i}\mu_i(P,K)\mu_{e-i}(Q,K),$$

where constants $c_{e,i}$ depending on p, q, n, e, and i are given by

$$(2) \qquad c_{e,i} = O_{n+1}O_n \cdots O_2 \frac{\frac{O_{p+q-n+1}O_{p+q-n+2}(e/2)!}{O_{p+1-n-e+2}}}{\left(\frac{O_{p+1}O_{p+2}(i/2)!}{O_{p-i+2}}\right)\left(\frac{O_{q+1}O_{q+2}((e-i)/2)!}{O_{q-e+i+2}}\right)}.$$

Here $O_j = 2\pi^{j/2}/\Gamma(j/2)$ is the volume of unit sphere in Euclidean j-space and the integral invariants $\mu_e(P,K)$, $0 \leq e$ even $\leq p$, are the integrals of I_e over P, where I_e are given by

$$(3) \quad I_e = \frac{(p-e)!}{2^{e/2}p!} \sum \delta\binom{\alpha}{\beta}(R_P - K)_{\alpha_1\alpha_2\beta_1\beta_2} \cdots (R_P - K)_{\alpha_{e-1}\alpha_e\beta_{e-1}\beta_e},$$

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while $\delta\binom{\alpha}{\beta}$ is $+1$ or -1 according as $\alpha_1, \ldots, \alpha_e$ are distinct and an even or odd permutation of $\beta_1, \ldots, \beta_e$, and otherwise $\delta\binom{\alpha}{\beta}$ is zero. The summation in I_e is taken over all α's and β's running from 1 to p. The integral invariants $\mu_e(P, K)$ are related to Weyl's curvature invariants $k_e(R_P - K)$ by

$$\mu_e(P,K) = \frac{2^{e/2}(p-e)!\,(e/2)!}{p!} k_e(R_P - K). \tag{4}$$

Chern's linear kinematic formula in Euclidean space [2] also holds for non-Euclidean space. In fact Ishihar [4] showed that for $P \subset E^n(K)$, $0 \leq e$ even $\leq p+q-n$,

$$\int \mu_e(P \cap E^q, K)\, dE^q = \frac{O_{n+1} \cdots O_{n-q+1} O_{p+q-n+2} O_{p+q-n+1} O_{p+2-e}}{O_{q+1} \cdots O_1 O_{p+2} O_{p+1} O_{p+q-n+2-e}} \mu_e(P, K), \tag{5}$$

where dE^q is the density for q-planes in $E^n(K)$.

In this paper we derive some consequences of (1) and (5). To be more specific we prove the following propositions related to integral geometry and tube volumes.

PROPOSITION 1. *For a pair of compact embedded submanifolds P and Q of an elliptic space $E^n(K)$, $K > 0$, there exists a motion $g \in E_K(n)$ such that for $p + q \geq n$*

$$\mu_e(P \cap gQ, K) \geq \frac{2K^{n/2}}{O_{n+1} \cdots O_2} \sum_i c_{e,i} \mu_i(P, K) \mu_{e-i}(Q, K), \tag{6}$$

where $c_{e,i}$ are given by (2).

PROPOSITION 2. *For $P \subset E^n(K)$, $K > 0$, there exists a motion $g \in E_K(n)$ such that for $p + q \geq n$*

$$\mu_e(P \cap gE^q, K) \geq K^{(n-q)/2} \mu_e(P, K) \tag{7}$$

$$\times \frac{O_{n-q} \cdots O_1 O_{n+1} \cdots O_{n-q+1} O_{p+q-n+2} O_{p+q-n+1} O_{p+2-e}}{O_{n+1} \cdots O_1 O_{p+2} O_{p+1} O_{p+q-n+2-e}}.$$

For $P \subset E^n(K)$ we denote by $V_P^{E^n(K)}(r)$ the n-dimensional volume of a solid tube $T(P, r)$ of radius r about P and by $A_P^{E^n(K)}(r)$ the $(n-1)$-dimensional volume of its boundary P_r. It is interesting to integrate $V_{P \cap gQ}^{E^n(K)}(r)$ and $V_{P \cap E^q}^{E^n(K)}(r)$ over dg and dE^q respectively.

PROPOSITION 3. (i) *For $P \subset E^n(K)$ we have*

$$\int V_{P\cap E^q}^{E^n(K)}(r)\, dE^q = \sum_{\substack{0 \le e \le p+q-n \\ e \text{ even}}} a_e k_e(R_P - K), \tag{8}$$

where constant a_e are given by (14) (*see* §2).

(ii) *For P, $Q \subset E^n(K)$ we have*

$$\int V_{P\cap gQ}^{E^n(K)}(r)\, dg = \sum_{\substack{0 \le e \le p+q-n \\ e \text{ even}}} \sum_{\substack{0 \le i \le e \\ i \text{ even}}} \alpha_e c_{e,i} \mu_i(P, K) \mu_{e-i}(Q, K), \tag{9}$$

where constants α_e and $c_{e,i}$ are given by (15) (*see* §2) *and* (2) *respectively.*

PROPOSITION 4. *For P, $Q \subset E^n(K)$ we have*

$$\int \mu_e((P \cap gQ)_r, K)\, dg \tag{10}$$

$$= \sum_{c=0}^{[(p+q-n)/2]} \sum_{k=0}^{p+q-n-2c} \sum_{\substack{0 \le i \le 2c \\ i \text{ even}}} \alpha_{k,c} c_{2c,i} \mu_i(P, K) \mu_{2c-i}(Q, K),$$

where $\alpha_{k,c}$ are given by (17) (*see* §2).

PROPOSITION 5. *Let $S_x^{n-1}(r)$ denote the $(n-1)$-dimensional geodesic sphere of radius r with the center x in $E^n(K)$. For an odd-dimensional*

$P \subset E^n(K)$ we have

$$\int_{T(P,r)} \mu_{p-1}(P \cap S_x^{n-1}(r), K)\, dx = \frac{2^{(p-1)/2}\pi^{(p-3)/2} A_P^{E^{n+2}(K)}(r)}{(p-2)(p-4)\cdots 3\cdot 1 \left(\frac{\sin\sqrt{K}r}{\sqrt{K}}\right)\cos\sqrt{K}r}. \tag{11}$$

2. Proof of Propositions

We refer to [6] for basic facts on non-Euclidean integral geometry and use the notations of [2, 6].

PROOF OF PROPOSITION 1: For an elliptic space $E^n(K)$, $K > 0$, the compact group $E_K(n) = E^n(K)\dot{\times} SO(n)$ of proper motions has its total measure $K^{-n/2}O_{n+1}\cdots O_2/2$ according to (12.35), (17.46) in [6] (see also (12) below). Therefore the mean value of $\mu_e(P \cap gQ, K)$ is equal to the right-hand side of (6).

PROOF OF PROPOSITION 2: From (17.53b) in [6] we have

$$\int dE^q = K^{(n-q)/2}\frac{O_{n+1}\cdots O_{q+2}}{O_{n-q}\cdots O_1}, \qquad K > 0. \tag{12}$$

The mean value of $\mu_e(P \cap E^q, K)$ is equal to the right-hand side of (7).

PROOF OF PROPOSITION 3: (i) Combining Weyl's tube formula for $P \subset E^n(K)$

$$A_P^{E^n(K)}(r) = \sum_{c=0}^{[p/2]} \frac{O_{n-p+2c}}{(2\pi)^c} k_{2c}(R_P - K)\left(\frac{\sin\sqrt{K}r}{\sqrt{K}}\right)^{n-p+2c-1}\left(\cos\sqrt{K}r\right)^{p-2c} \tag{13}$$

with the linear kinematic formula (5) we obtain (8) by a strightforward computation, where the constants a_e are given by

$$a_e = \frac{\pi^{(2n-p-q)/2}O_{n+1}\cdots O_{n-q+1}O_{p+q-n+2}O_{p+q-n+1}O_{p+2-e}}{2^{-1+e/2}\Gamma\left(\frac{2n-p-q+e}{2}\right)O_{q+1}\cdots O_1 O_{p+2}O_{p+1}O_{p+q-n+2-e}} \times \int_0^r \left(\frac{\sin\sqrt{K}s}{\sqrt{K}}\right)^{2n-p-q+e-1}(\cos\sqrt{K}s)^{p+q-n-e}\, ds. \tag{14}$$

(ii) Similarly from (13) and (1) we get (9), where the constants α_e are given by

$$\alpha_e = \frac{\pi^{(2n-p-q)/2}(p+q-n)!}{2^{e-1}\Gamma\left(\frac{2n-p-q+e}{2}\right)\left(\frac{e}{2}\right)!\,(p+q-n-e)!} \times \int_0^r \left(\frac{\sin\sqrt{K}s}{\sqrt{K}}\right)^{2n-p-q+e-1} (\cos\sqrt{K}s)^{p+q-n-e}\, ds. \tag{15}$$

PROOF OF PROPOSITION 4: From the Steiner formulas and the expressions of the integrated mean curvatures of the tubular hypersurface P_r in [1, 3] we can find that for $P \subset E^n(K)$

$$\begin{aligned} &k_{2\ell}(R_{P_r} - K) \\ &= \sum_{c=0}^{[p/2]} \sum_{k=0}^{p-2c} \frac{2^\ell \Gamma\left(\frac{1}{2}+\ell\right)\binom{n-p-1+2c}{2\ell-k}\binom{p-2c}{k}(-K)^k k_{2c}(R_P - K)}{\pi^{\frac{1}{2}}(n-p)\cdots(n-p+2c-2)} \\ &\times \left(\frac{\sin\sqrt{K}r}{\sqrt{K}}\right)^{n-p-1+2c+2k-2\ell} (\cos\sqrt{K}r)^{p-2c+2\ell-2k}. \end{aligned} \tag{16}$$

Hence from (1) and (16) we obtain (10) where $\alpha_{k,c}$ are given by

$$\begin{aligned} \alpha_{k,c} &= \frac{2^{3e/2-c}}{\pi^{1/2}} \\ &\times \frac{(n-1-e)!\,\left(\frac{e}{2}\right)!\,(p+q-n)!\,\Gamma\left(\frac{1+e}{2}\right)\binom{2n-p-q-1+2c}{e-k}\binom{p+q-n-2c}{k}}{(n-1)!\,(p+q-n-2c)!\,c!\,(2n-p-q)\cdots(2n-p-q+2c-2)} \\ &\times \left(\frac{\sin\sqrt{K}r}{\sqrt{K}}\right)^{2n-p-q-1+2c+2k-e} (\cos\sqrt{K}r)^{p+q-n-2c+e-2k}(-K)^k. \end{aligned} \tag{17}$$

PROOF OF PROPOSITION 5: Let P be a compact odd-dimensional manifold embedded in $E^n(K)$. We will apply Chern's kinematic formula in noneuclidean space with P as the stationary submanifold and with $S^{n-1}(r)$ as the moving submanifold of $E^n(K)$. Here $S^{n-1}(r)$ is the

$(n-1)$-dimensional geodesic sphere of radius r in $E^n(K)$. Let x be the center of $gS^{n-1}(r)$, $g \in E_K(n)$. Since $E_K(n) = E^n(K)\dot{\times}SO(n)$ we can write $gS^{n-1}(r) = g_0S_x^{n-1}(r)$, where $g_0 \in SO(n)$ and $S_x^{n-1}(r)$ denotes the $(n-1)$-dimensional geodesic sphere of radius r with the center x. We find that for odd $p \geq 1$

$$\begin{aligned}
&\int \mu_{p-1}(P \cap gS^{n-1}(r), K)\, dg \\
&= \int_{E^n(K)} \left\{ \int_{SO(n)} \mu_{p-1}(P \cap g_0 S_x^{n-1}(r), K)\, dg_0 \right\} dx \\
&= \int_{SO(n)} \left\{ \int_{T(P,r)} \mu_{p-1}(P \cap g_0 S_x^{n-1}(r), K)\, dx \right\} dg_0 \\
&= O_n \cdots O_2 \int_{T(P,r)} \mu_{p-1}(P \cap S_x^{n-1}(r), K)\, dx,
\end{aligned}$$

where dx is the volume element of $E^n(K)$. Therefore from Chern's kinematic formula (1) in non-Euclidean space and from

$$c_{p-1,i} = \frac{p!\, \pi^{(n+1)/2} 2^{(p-2i+3)/2} O_{n-1} \cdots O_3}{(i/2)!\,(p-i)!\,(p-2)(p-4)\cdots 3\cdot 1\Gamma((n-p+i+2)/2)}$$

we obtain

$$\begin{aligned}
&O_n \cdots O_2 \int_{T(p,r)} \mu_{p-1}(P \cap S_x^{n-1}(r), K)\, dx \\
&= \sum_{\substack{0 \leq i \leq p-1 \\ i \text{ even}}} c_{p-1,i}\mu_i(P,K)\mu_{p-i-1}(S^{n-1}(r),K) \\
&= \sum_{\substack{0 \leq i \leq p-1 \\ i \text{ even}}} \frac{2^{\frac{p+1}{2}} \pi^{\frac{n-1}{2}} O_{n-1} \cdots O_2 k_i(R_P - K)}{(p-2)(p-4)\cdots 3\cdot 1\cdot 2^{\frac{i}{2}}\Gamma\left(\frac{n+2-p+i}{2}\right)} \left(\frac{\sin\sqrt{K}r}{\sqrt{K}}\right)^{n-p+i} \\
&\times (\cos\sqrt{K}r)^{p-i-1} O_n
\end{aligned}$$

$$= O_n \cdots O_2 \frac{2^{\frac{p-1}{2}} \pi^{\frac{p-3}{2}}}{(p-2)(p-4)\cdots 3 \cdot 1 \left(\frac{\sin\sqrt{K}r}{\sqrt{K}}\right) \ldots \sqrt{K}r}$$

$$\times \sum_{\substack{0 \le i \le p-1 \\ i \text{ even}}} \frac{\pi^{\frac{n+2-p}{2}} k_i(R_P - K)}{2^{\frac{i}{2}-1}\Gamma\left(\frac{n+2-p}{2} + \frac{i}{2}\right)} \left(\frac{\sin\sqrt{K}r}{\sqrt{K}}\right)^{n+2-p+i-1} (\cos\sqrt{K}r)^{p-i}.$$

Finally we get the desired formula from (13).

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