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TORSION THEORY AND LOCAL COHOMOLOGY

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Let R be a commutative ring with 1 and let σ be a half–centered idempotent kernel functor on the category of R–modules with the set F of all primes ideals p of R such that R/p is σ–torsion free. As a generalization of Theorem 6.1 [5] we prove that for an R–module M, the localization $Q_\sigma(M)$ is isomorphic to $\Gamma(F, \tilde{M})$, where $\tilde{M}$ is the sheaf on Spec(R) associated to M.

Using this result, we prove the similar result that finitely presented R–module is projective iff it is locally free holds with respect to σ.

1. Preliminaries

Let R be a ring and R–mod be the category of all (left) R–modules. A functor σ from R–mod to itself is called an idempotent kernel functor or torsion radical if it has the following property:

(1) $\sigma(M)$ is a submodule of M.
(2) If $f : M \to N$ is a homomorphism then $f(\sigma(M)) \subset \sigma(N)$ and $\sigma(f)$ is the restriction of f to $\sigma(M)$.
(3) If N is a submodule of M, $\sigma(N) = N \cap \sigma(M)$.
(4) $M/\sigma(M)$ is σ–torsion free i.e. $\sigma(M/\sigma(M)) = 0$.

We say M is σ–torsion if $\sigma(M) = M$.

The equivalent concepts of idempotent kernel functors [9], torsion radicals and Gabriel topologies [4,12] were originally investigated to extend localization techniques in commutative rings to the case of non–commutative rings. However there are intersting questions even in the context of commutative rings.

In this paper we assume all the rings are commutative with identity. For an idempotent kernel functor σ, let $L(\sigma)$ denote the associated

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Gabriel topology which consists of ideals I of R with the property that R/I is σ–torsion. σ then partitions set Spec(A) of all the prime ideals of R into two sets T and F, where T is the set of all primes such that R/p is σ–torsion, F is the set of all primes such that R/p is σ–torsion free.

Given any idempotent kernel functor σ, the associated localization functor Q (-) is defined as follows:

R–module E is σ–injective if it has the following property: if M is any module and N is a submodule of M such that $\sigma(M/N) = M/N$, then any R–homomorphism from N to E extends to a homomorphism from M to E. E is called faithfully σ–injective if, in the above, the homomorphism from N to E has a unique extension to M. R–module E is σ–faithfully injective if and only if E is σ–torsion free and E is σ–injective. For σ–torsion free module M, there is a unique faithfully σ–injective module up to isomorphism which contains M and such that E/M is σ–torsion [9]. This unique faithfully σ–injective module is denoted by Q (M). For any R–module M, Q (M) is defined by Q $(M/\sigma(M))$.

We mention following 3 examples:

(1) For $p \in$ Spec(R), let $L(p)$ be the set of ideals of R which are not in p. Let σ be the idempotent kernel functor determined by $L(p)$. Then σ is the usual localization functor, i.e. for any R–module M, σ $(M) = \{m \in M \mid m_p = 0 \text{ in } M_p\}$ and Q $(M) = M_p$.

(2) Let P be a set of prime ideals of R. Define

$$\begin{aligned} L(P) &= \{I \lhd R \mid I_p = R_p \text{ for all } p \in P\} \\ &= \cap_{p \in P} L(p) \end{aligned}$$

Then $L(P)$ is a Gabriel topology and this defines an unique idempotent kernel functor.

In particular if Z is the set of height one prime ideals of krull domain R, then $L(Z) = \{I \lhd R \mid I_p = R_p \text{ for all } p \in Z\}$. For a torsion free R–module M in the usual sense, $Q(M) = \cap_{p \in Z} M_p$, where M is considered as a submodule of $M \otimes K$ (K is the field of quotients of R).

(3) Let I be an ideal and $L(I)$ be the Gabriel topology generated by $\{I^n \mid n \geq 1\}$. Then we have $T = \{p \in \text{Spec}(R) | I \subset P\}$, $F = \{p \in \text{Spec}(R) \mid I \not\subset P\}$.

In the next section we consider the localization functor of this idempotent kernel functor in (3).

Let Ass(M) be the set of primes p, where p is the minimal among the primes containing annihilator ann(x) of non–zero element x of M.

Let σ be an idempotent kernel functor which corresponds to the partition (T, F) of Spec(R). Then the following holds [4]

M is σ–torsion $\Rightarrow$ Ass(M) $\subset T$

Ass(M) $\subset F \not\Rightarrow M$ is σ–torsion free.

From [4] we recall the following:

σ is half–centered if Ass(M) $\subset T \Longleftrightarrow M$ is σ–torsion.

σ is well–centered if σ is half–centered and Ass(M) $\subset F \Longleftrightarrow M$ is σ–torsion free.

It is clear that σ is half–centered if and only if the partition (T, F) of Spec(R) defined by σ defines a unique idempotent kernel functor. This unique idempotent kernel functor is defined as in (2).

We say an idempotent kernel functor σ is of finite type if every ideal I in the Gabriel topology $L(\sigma)$ associated to σ contains a finitely generated ideal which is also in $L(\sigma)$.

In [12] it is shown that if σ is an idempotent kernel functor of finite type then it is well–centered.

Note. Let R be a krull domain and Z be the set of height one prime ideals of R. Then $L(Z)$ is of finite type and hence the idempotent kernel functor is well–centered.

For any $I \in L(Z)$, there is $a(q) \in I\backslash q$ for some $q \in Z$. Since $a(q)$ is contained only finitely many primes in p_1, $p_2, \ldots, p_n$ in Z, choose $a(p_i) \in I\backslash p_i$. Then $(a(q), a(p_1), \ldots, a(p_n)) \not\subset p$ for all p in Z. Hence $L(Z)$ is of finite type.

For further definitions and properties, we refer [6, 13].

2. Torsion radical and Quotient functor

Let σ be an idempotent kernel functor on R–mod and (T, F) be the corresponding partition of Spec(R). Let $\sigma_1, \ldots, \sigma_n, \ldots$ be the derived functors of σ.

As a generalization of Lemma 5.2 and proposition 5.1 [5] we have the following:

LEMMA 1.1. *Let σ be half-centered. If M is σ-torsion then*

$$\sigma_n(M) = 0 \quad \textit{for all} \quad n > 0.$$

Proof. The proof of Lemma 5.2 also holds if σ is half-centered.

PROPOSITION 2.2. *Let σ be a half-centered idempotent kernel functor on R-mod and M be any R-mod. Then there is an exact sequence*

$$0 \to \sigma(M) \to M \to Q(M) \to \sigma_1(M) \to 0$$

We note that M is faithfully σ-injective i.e., $M = Q(M)$ if and only if $\sigma(M) = \sigma_1(M) = 0$.

We now consider the relation between the derived functors of σ and the local cohomology group functors. For any R-mod M, let $\tilde{M}$ be the associated quasicoherent sheaf on Spec(R).

Let Z be any closed set in $X =$ Spec(R) and let M be a R-module. Then $\Gamma_Z(X, \tilde{M})$ is the submodule of M consisting of all those sections of $\tilde{M}$ whose support is contained in Z. Let $H^n_Z(X, \cdot)$ be the derived functors of $\Gamma_Z(X, \cdot)$. Then we have the following proposition.

PROPOSITION 2.3. *Let σ be a half-centered idempotent kernel functor on R-mod which corresponds to a partition (T, F) of Spec$(R) = X$ such that T is a closed set in X. Then for any R-module M, $\sigma_n(M) \cong H^n_T(X, \tilde{M})$ for all n.*

Proof. By the definition, $\Gamma_Z(X, \tilde{M})$ is the largest submodule of M whose support is in T. Since σ is half-centered, $\Gamma_Z(X, \tilde{M}) \cong \sigma(M)$. Hence for all n, $\sigma_n(M) \cong H^n_T(X, M)$.

PROPOSITION 2.4. *Let σ be a half-centered idempotent kernel functor on R-mod which corresponds to a partition (T, F) of Spec$(R) = X$ such that T is a closed set in X. Then for any R-module M there is an exact sequence*

$$0 \to \Gamma_T(X, \tilde{M}) \to \Gamma(X, \tilde{M}) \to \Gamma(F, \tilde{M}) \to H^1_T(X, \tilde{M}) \to 0$$

and

$$0 \to \sigma(M) \to M \to \Gamma(F, \tilde{M}) \to \sigma_1(M) \to 0$$

Proof. Since T is closed, there is an exact sequence

$$0 \to \Gamma_T(X, \tilde{M}) \to \Gamma(X, \tilde{M}) \to \Gamma(F, \tilde{M}) \to H^1_T(X, \tilde{M}) \to H^1(X, \tilde{M}) \to$$

by corollary 1.9 [8]. Since X is an affine scheme and $\tilde{M}$ is quasi-coherent, $H(X, \tilde{M}) = 0$ by Theorem 9.8 [10].

The second exact sequence follows from the first by Proposition 2.3.

Let σ be an idempotent kernel functor on R-mod which corresponds to the partition (T, F) of Spec(R). Let S be a multiplicative set in R. Then $S^{-1}\sigma$ is the idempotent kernel functor on $S^{-1}R$-mod induced by the canonical morphism $\varphi : R \to S^{-1}R$, i.e. $L(S^{-1}(\sigma)) = \{J \triangleleft S^{-1}R \mid \varphi^{-1}(J) \in L(\sigma)\}$.

For any injective R-module I, $S^{-1}I$ is an injective $S^{-1}R$-module, we have the following proposition as a generalization of Proposition 3.2 [5].

PROPOSITION 2.5. *Let σ be a half-centered idempotent kernel functor on R-mod and $\sigma_1, \ldots, \sigma_n, \ldots$ be its derived funtors. Let S be a multiplicative set of R and $S^{-1}\sigma_1, \ldots, S^{-1}\sigma_n, \ldots$ be the derived functors of $S^{-1}\sigma$. Then for any R-module M, $S^{-1}(\sigma_n(M)) \cong \sigma_n(S^{-1}M) \cong (S^{-1}\sigma_n)(S^{-1}M)$ for all n.*

As a generalization of Theorem 6.1 [5] we have the following:

THEOREM 2.6. *Let σ be a half-centered idempotent kernel functor which corresponds to the partition (T, F) of Spec(R). Assume F is a quasi-compact open set in X = Spec(R). Then for any R-module M, the map $j : M \to \Gamma(F, \tilde{M})$ can be identified with the canonical map $M \to Q(M)$, i.e., $Q(M) \cong \Gamma(F, \tilde{M})$.*

Proof. By Proposition 2.2 and 2.4 and 2.5, the same proof holds as in [5].

COROLLARY 2.7. *Let M and N be R-modules such that $\tilde{M}(F) = \tilde{N}(F)$ in the theorem. Then $Q(M) = Q(N)$.*

In the following we consider some local properties of R-modules with respect to σ which are useful in the study of the Brauer groups relative to the torsion theory [3, 11].

Let σ be an idempotent kernel functor on R–mod and (T, F) be the partition of Spec(R) associated to σ. We denote by $C(\sigma)$ the set of all idelas of R which are maximal with respect to the property of not being contained in $L(\sigma)$. Then $C(\sigma) \subset F$. We recall the following definitions [3].

DEFINITION. *Let M be an R–module.*

(1) *M is σ–finitely generated if there exists a finitely generated R–module N such that $N \subset M$ and $Q(M) = Q(N)$.*
(2) *M is σ–finitely presented if there exists an R–homomorphism $f : N \to M$ such that N is finitely presented,* ker f *and coker* f *are σ–torsion.*
(3) *M is σ–faithful if M is σ–closed and faithful as a $Q(R)$–module.*
(4) *M is σ–quasiprojective if M_p is free R_p–module for all $p \in C(\sigma)$.*
(5) *R is σ–noetherian if every ideal I in $L(\sigma)$ is σ–finitely generated.*

In (4), M_p is R_p–free for all p in $C(\sigma)$ if and only if M_p is R_p–free for all p in F.

For the consideration of right exactness of Q, the concept of σ–projectiveness is defined. R–module P is σ–projective if the following holds [9].

Given σ–torsion free modules M' and M and an epimorphism $M' \to M \to 0$ and given a homomorphism $P \to M$, there is a submodule P' of P with P/P' σ–torsion, and a homomorphism $P' \to M'$ such that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & P' & \longrightarrow & P & & \\ & & \downarrow & & \downarrow & & \\ & & M' & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

is commutative.

In the above definition we say P is weak σ–projective if we further assume that M' and M are faithfully σ–injective.

Note. Let P be an R–module. Then the following hold:
(1) If P is projective in the usual sense then P is σ–projective.
(2) If P is σ–projective then P is weak σ–projective.

(3) If P is weak σ–projective then P is σ–quasiprojective.

For a finitely presented R–module P, it is well known that P is projective if and only if P_q is free R_q–module for all q in Spec(R). It is not clear whether the similar result holds with respect to an idempotent kernel fucntor σ i.e. for a σ–finitely presented R–module P, P is σ–projective (or weak σ–projective) if and only if P_q is free R_q–module for all q in $C(\sigma)$.

We note that a sheaf of O_X–module on a scheme $(X, 0_X)$ is locally free of finite type if it is finitely presented and for all $x \in X$, the stalk is a free $O_{X,x}$–module.

PROPOSITION 2.8. *Let σ be an idempotent kernel functor on R–mod which corresponds to the partition (T, F) of Spec(R). We further assume that R is σ–noetherian. If F is a quasi–compact open set in X there is a one–to–one correspondence among the following classes*

(A) *locally free sheaves of finite type over F*
(B) *faithfully σ–injective, weak σ–projective and σ–finitely generated R–modules*
(C) *faithfully σ–injective σ–quasiprojective and σ–finitely generated R–modules.*

Proof. Let X be Spec(R) and let m be a locally free sheaf of finite type over F. let $i_*(m)$ be the direct image of m on X induced by the canonical inclusion $i : F \to X$. Since F is quasi–compact, F can be covered by finitely many special open sets $D(f_1), \ldots, D(f_n)$. By the sheaf property there is an exact sequence of sheaves on X

$$0 \to i_*(m) \to \oplus_i i_*(m|D(f_i)) \to \oplus_{i,j} i_*(m|D(f_i f_j))$$

By 5.2(d) [8] $i(m_x|D(f_i))$ and $i(m|D(f_i f_j))$ are quasi–coherent, $i_*(m)$ is also quasi–coherent by 5.7. [8]. Hence $i_*(m) = \tilde{M}$ for some R–module M. By the definition of m and $i_*(m)$, $m(D(f_i)) = M_{f_i}$ is finitely generated R_{f_i}–module for each $i = 1, \ldots, n$. We may assume M_{f_i} is generated by $\{\frac{x_{i1}}{1}, \ldots, \frac{x_{in_i}}{1}\}$.

Let N be a submodule of M generated by the set $\{x_{ij} \mid 1 \leq i \leq n,\ 1 \leq j \leq n_i\}$. Then $N_{f_i} = M_{f_i}$ and hence $\Gamma(F, \tilde{N}) = \Gamma(F, \tilde{M})$. By Corollary

2.7, $Q(N) = Q(M)$ and $Q(M)$ is faithfully σ–injective and σ–finitely generated. Given epimorphism $A \to B$ of faithfully σ–injective modules and homomorphism $Q(M) \to B$, there is a commutative diagram

$$\begin{array}{ccccc} & & Q(M) & & \\ & \swarrow & \downarrow & & \\ A & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

since $\tilde{M}(F)$ $(= Q(M))$ is a projective $\tilde{R}(F)$ $(= Q(R))$–module. Hence $m \mapsto Q(M)$ is a well–defined one–to–one map from the set of locally free sheaves of finite type over F to the set of faithfully σ–injective, weak σ–projective and σ–finitely generated R–modules. It is clear that every weak σ–projective module is σ–quasiprojective.

Let M be a faithfully σ–injective, σ–quasi–projective and σ–finitely generated R–module. Since R is σ–noetherian, M is σ–finitely presented by proposition [11] and hence there is a finitely presented R–module N such that $Q(N) \cong Q(M)$ $(= M)$. For each p in F, $N_p \cong Q(N)_p \cong M_p$, $\tilde{N}|_F$ is a locally free sheaf of finite type. By Corollary 2.7, the map $M \to \tilde{N}|_F$ is a well–defined one–to–one map from the set of faithfully σ–injective, σ–quasiprojective and σ–finitely generated R–modules to the set of locally free sheaves of finite type over F.

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