Comm. Korean Math. Soc. 4 (1989), No. 2, pp. 273~277

REMARKS ON TIGHT CLOSURE OVER COHEN–MACAULAY LOCAL RING

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0. Introduction

Recently M.Hochster and C.Huneke developed the theory about tight closure to explore other related topics such as invariant theory, monomial conjecture and the syzygy theorem. (ref. [1], [2], [3], [4]).

In this paper, using the notion of tight closure we prove that the homomorphic image of the Cohen–Macaulay (C–M) local ring is C–M under certain condition in Theorem 2.1.

1. Preliminaries

Throughout this paper, all rings are noetherian and commutative with 1. We introduce the notion of the tight closure of an ideal I in R, when $\operatorname{Char} R = p$, p is a positive prime number. Let $R^0 = R - \cup\{P : P \text{ is a minimal prime of } R\}$.

DEFINITION 1.1. *Let $I \subseteq R$ be an ideal. If $\operatorname{Char} R = p > 0$, we say that $x \in R$ is in the tight closure, I^*, of I, if there exists $c \in R^0$ such that for all $e \gg 0$, $cx^{p^e} \in I^{[p^e]}$, where $I^{[q]} = (i^q : i \in I)$ when $q = p^e$. If $I = I^*$, we say that I is tightly closed.*

REMARKS 1.2. *(1) If R is regular, then $I = I^*$ for all I. (ref. [1], [2], [3], [4]). (2) A Gorenstein local ring has the property that $I = I^*$ for all I if and only if the ideal generated by a single system of parameters is tightly closed. (ref. [1].)*

THEOREM 1.3 ([5]). *Let $P_1, \cdots, P_n$ be prime ideals in a commutative ring R, let I be an ideal in R, and x an element of R such that $(x, I) \not\subseteq$*

Received April 13, 1989.

$P_1 \cup \cdots \cup P_n$. *Then there exists an element* $i \in I$ *such that* $x + i \notin P_1 \cup \cdots \cup P_n$.

DEFINITION 1.4. *Let* $P_1, \cdots, P_n$ *be all minimal prime ideals in* R. *If* $\dim R/P_i = \dim R/P_j$ *for every* $i, j = 1, \cdots, n$, *we say that* R *is equidimensional.*

LEMMA 1.5. *Let* $R = S/I$ *where* S *is a C–M local ring and assume* R *is equidimensional. Let* $\{Q_1, \cdots, Q_n\}$ *be the minimal primes over* I. *Assume that* $x_1, \cdots, x_d$ *are parameters of* R. *Then there exist elements* $z_1, \cdots, z_h$ *in* I *and* $y_1, \cdots, y_d$ *such that the* y_i *lift* x_i, *the* z'*s and* y'*s together form a regular sequence, and there exists a* $c \notin \cup_{i=1}^{n} Q_i$ *and an integer* $q = p^e$ *such that* $cI^{[q]} \subset (z_1, \cdots, z_h)$, *where* $htI = h$.

Proof. First lift the elements $x_i \in R$ to elements y_i in S inductively in such a way that for every i, $0 \le i \le d-1$, y_{i+1} is not in any minimal prime of an ideal generated by a subset of $\{y_1, \cdots, y_i\}$. Choose $y \in S$ such that $y + I = \overline{y} = x_{i+1}$ in R.

If there is no lifting of x_{i+1} satisfying the previous condition, $Sy + I$ is contained in a minimal prime P of $(y_1, \cdots, y_m)S$, where $m \le i$ by Theorem 1.3. Since $\overline{y_1} = x_1, \cdots, \overline{y_m} = x_m$, $\overline{y} = x_{i+1}$, where $\overline{y_j}$ are the images of y_j in R for $j = 1, \cdots, m$, and S is C–M, $htP \ge m+1$. But P is a minimal prime ideal over $(y_1, \cdots, y_m)S$, and so $htP = m$, a contradiction.

Second, choose $z_1, \cdots, z_h \in I$ inductively such that for every j, $0 \le j \le h-1$, $z_{j+1} \in I - \cup P$, P is the minimal prime ideals of the ideal generated by $(z_1, \cdots, z_j)S$ and a subset of the y_i's. If this were impossible, for some subset of the y_i's, say, $\{y_1, \cdots, y_t\}$, $t \le d$, it would follow that there exists a minimal prime ideal P of $(z_1, \cdots, z_j)S + (y_1, \cdots, y_t)S$ that contains I by Theorem 1.3. Working in S_p, $ht(I + (y_1, \cdots, y_t) \ge htI + t = h + t$ since S_p is also C–M, and $\overline{y_1} = x_1, \cdots, \overline{y_t} = x_t$. But $htP = j + t < h + t$ since P is a minimal prime of $(z_1, \cdots, z_j)S + (y_1, \cdots, y_t)S$ and S is C–M, a contradiction.

Third, choose c and $q = p^e$ by the induction on htI. Assume $htI = 0$. Then every Q_i is a minimal prime ideal since R is equidimensional and S is C–M.

Let N be the nilradical of S. If $\{Q_i\}$ is the family of all minimal prime ideals of S, then $\cap Q_i = N$. Since $\operatorname{Char} S = p > 0$ is a positive prime number and N is finitely generated, there is an integer $q = p^e$ such that $N^{[q]} = 0$. So $I^{[q]} = 0$ since $I^{[q]} \subset N^{[q]} = 0$. Hence we can choose $c = 1$, and $cI^{[q]} = I^{[q]} = 0$. If there is a minimal prime ideal P with $P \not\supset I$, say, $P_1, \cdots, P_m$, then we can choose a $c' \in \cap_{j=1}^m P_j - \cup_{i=1}^n Q_i$. But $c'I \subset N$ and so $(c')^q I^{[q]} \subseteq N^{[q]} = 0$ for some $q = p^e$. Hence we can also choose $c = (c')^q \notin \cup_{i=1}^n Q_i$ and $cI^{[q]} = (c')^q I^{[q]} = 0$. Thus we may assume that $htI \geq 1$. Let $\overline{S} = S/z_1S$, $\overline{I} = I/z_1S$, $\overline{Q_j} = Q_j/z_1S$ and $\overline{z}_2, \cdots, \overline{z}_h$ be the images of $z_2, \cdots, z_h$ in $\overline{S}$. Then $\overline{S}/\overline{I} = S/I = R$ and $ht\overline{I} < htI$. Since $\overline{S}$ is also C–M, there is an element $\overline{c} \notin \cup_{i=1}^n \overline{Q_i}$, where $\overline{c}$ is the image of c in S and an integer $q = p^e$ such that $\overline{c}(\overline{I})^{[q]} \subset (\overline{z}_2, \cdots, \overline{z}_h)$. Therefore, $cI^{[q]} \subset (z_1, \cdots, z_h)S$, as desired.

THEOREM 1.6. *Let $R = S/I$ be equidimensional where S is a C–M local ring. Assume that* $\operatorname{Char}(R) = p > 0$. *Let $x_1, \cdots, x_n$ be elements of R which are parts of a system of parameters. Let $J = (x_1, \cdots, x_{n-1})R$. Then $J :_R x_n \subset J^*$.*

Proof. Let $a' \in J :_R x_n$. Let c, z_i, y_j, and q' be as in Lemma 1.5, and let a be a lifting of a' to S. Since $a'x_n \in J$, $ay_n \in (y_1, \cdots, y_{n-1}) + I$. Then for all $q = p^e \geq q'$, we obtain that $a^q y_n^q \in (y_1, \cdots, y_{n-1})^{[q]} + I^{[q]}$. Multiply by c and use Lemma 1.5 to get $ca^q y_n^q \in (y_1^q, \cdot, y_{n-1}^q, z_1, \cdot, z_h)$. As the $z's$ and $y's$ together form a regular sequence in S, this forces $ca^q \in (y_1^q, \cdots, y_{n-1}^q, z_1, \cdots, z_h)$. Reading modulo I gives that $c'a'^q \in J^{[q]}$ where c' is the image of c in R and is in R^0 by Lemma 1.5. Hence $a' \in J^*$.

2. Main Results

THEOREM 2.1. *If R is a one dimensional local ring which is the homomorphic image of a C–M ring, and there exists a system of parameter $x \in R$ such that (x) is tightly closed, then R is C–M.*

Proof. Let $R = S/I$ where S is a C–M local ring and I be an ideal in S.

The proof is by induction on $ht(I) = h$. It suffices to show that if $ht(I) = 0$, then R is C–M. In fact, if $ht(I) = h$ is positive, then there exists an element $z \in I$ which is not a zero divisor, since S is C–M. Let $\overline{S} = S/zS$ and $\overline{I} = I/zS$. Then $\overline{S}/\overline{I} = S/I = R$. Since z is not a zero divisor in S, $\overline{S}$ is also C–M and $ht(\overline{I}) = ht(I) - 1 < ht(I)$. Hence R is C–M by induction hypothesis.

Assume that $ht(I) = 0$. Let c, y and q' be as in Lemma 1.5, i.e., $cI^{[q']} = 0$ and $y + I = x$ in R. Suppose $x_1 x = 0$. Choose $y_1 \in S$ such that $x_1 = y_1 + I$ in R. Then $yy_1 \in I$ and $c(yy_1)^q \in cI^{[q]} = 0$ for all $q \geq q'$. Hence $cy_1^q = 0$ since y is not a zero divisor in S and hence $\overline{c}x_1^q = 0 \in (x)^{[q]}$ for all $q \geq q'$, where $\overline{c}$ is the image of c in R, in fact, $\overline{c} \in R^0$. It follows that $x_1 \in (x)^* = (x)$, where $(x)^*$ is the tight closure of (x). Thus $x_1 = xx_2$ for some $x_2 \in R$. By the similar method, we can see that x_1 is divided by x infinitely many times. This means that x_1 is 0 by a Krull intersection theorem and the Nakayama's Lemma. Therefore, x is not a zero divisor and R is C–M.

COROLLARY 2.2. *If R is an equidimensional local ring with* $\dim R = 2$ *which is the homomorphic image of a C–M local ring and if x and y are a system of parameters of R which satisfies (x) is tightly closed. Then R is C–M.*

Proof. By the similar method of the above proof, we see that x is not a zero divisor since (x) is tightly closed and R is equidimensional. On the other hand, $(y) : (x) \subset (x)^* = (x)$ by Theorem 1.6. Hence y is not a zero divisor of R/xR. Thus x and y are a regular sequence in R and so R is C–M.

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