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THE HYPERBOLIC METRIC IN CONVEX REGIONS *

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1. Introduction

Let X be a hyperbolic region in the complex plane and λ_X the density of the hyperbolic metric on X. Set $\delta_X(z) = \mathrm{dist}(z, \partial X)$; this is the distance from z to the boundary of X. There are few results that deal with the size of the hyperbolic density. The upper bound $\lambda_X(z) \leq 2/\delta_X(z)$ is a direct consequence of Schwarz' Lemma [5, p.45]. For an arbitrary hyperbolic region X there does not exist a positive number $c = c(X)$ such that $c/\delta_X \leq \lambda_X$. For instance, if $X = \{z : 0 < |z| < 1\}$, then

$$\lambda_X(z) = \frac{1}{|z|\log(1/|z|)}$$

and $\delta_X(z) = |z|$ for $0 < |z| \leq \frac{1}{2}$ so that $\lambda_X(z)\delta_X(z) \to 0$ as $z \to 0$. Minda [8] proved that if X is a proper convex region in the complex plane, then for $z \in X$

$$(1) \qquad 1/\delta_X(z) \leq \lambda_X(z)$$

with equality if and only if X is a half–plane.

Osgood [10] gave an estimate for the gradient of the logarithm of the hyperbolic metric of a simply connected region in terms of the hyperbolic density:

$$|\nabla \log \lambda_X(z)| \leq 2\lambda_X(z).$$

Minda [9] obtained a refinement of this inequality for proper convex regions. Namely,

$$(2) \qquad |\nabla \log \lambda_X(z)| \leq \lambda_X(z)$$

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with equality if and only if X is a half–plane.

In this paper we show that each of the inequalities (1) and (2) actually characterizes convex regions.

2. The hyperbolic metric

We begin this section with a short introduction to the hyperbolic metric. A general discussion of this subject can be found in [7]. Suppose X is a hyperbolic region in the complex plane $\mathbf{C}$; that is, X is a region in $\mathbf{C}$ such that $\mathbf{C} - X$ contains at least two points. Then by the General Riemann Mapping Theorem [1, p.142], there exists a holomorphic universal covering projection f of the open unit disk D onto X. The set of all such covering projections is given by $f \circ T$, where T ranges over all conformal automorphisms of D. The hyperbolic metric on D is the Riemannian metric

$$\lambda_D(z)|dz| = \frac{2|dz|}{1-|z|^2}.$$

The density λ_X of the hyperbolic metric is determined from

$$\lambda_X(f(z))|f'(z)| = \frac{2}{1-|z|^2},$$

where $f : D \to X$ is any covering. This is independent of the choice of the covering since

$$\frac{2|T'(z)|}{1-|T(z)|^2} = \frac{2}{1-|z|^2}$$

for any conformal automorphism T of D. Also, the hyperbolic metric is invariant under conformal mappings; if $g : X \to Y$ is a conformal mapping, then $\lambda_Y(g(z))|g'(z)| = \lambda_X(z)$. We will make use of the following property for the hyperbolic metric.

Principle of the Hyperbolic Metric. Suppose X and Y are hyperbolic regions. If f is holomorphic on X and $f(X) \subset Y$, then $\lambda_Y(f(z))|f'(z)| \leq \lambda_X(z)$ for each point z in X. Equality occurs at some point if and only if f is a holomorphic covering map.

Keogh [4] proved that if a region X is not convex, then there exists a holomorphic function $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ such that $f(D) \subset X$ but $\sigma_f(D) - X \neq \phi$, where $\sigma_f(z) = a_0 + \frac{1}{2}a_1 z$. We now show that the inequality (1) is a sufficient condition for a hyperbolic region to be convex.

THEOREM 1. *If X is a hyperbolic region in the complex plane such that $1/\delta_X(z) \le \lambda_X(z)$ for all z in X, then X is convex.*

Proof. It will be shown that X is convex by an application of Keogh's theorem. Suppose $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ is a holomorphic function such that $f(D) \subset X$. Then, by the Principle of the Hyperbolic Metric, we have $\lambda_X(a_0)|a_1| \le 2$. Set $\sigma_f(z) = a_0 + \frac{1}{2}a_1 z$. Then

$$\begin{aligned}|\sigma_f(z) - a_0| &= \frac{1}{2}|a_1||z| < \frac{1}{2}|a_1| \\ &\le \frac{1}{\lambda_X(a_0)} \\ &\le \delta_X(a_0)\end{aligned}$$

for all z in D. This yields $\sigma_f(D) \subset X$, since $\delta_X(a_0)$ is the radius of the largest disk in X with center a_0. Therefore, Keogh's theorem gives that X is convex.

Minda [9] used some properties of euclidean and hyperbolic curvature to establish the inequality (2) for convex regions. Here we obtain the same result, but our method of proof is different.

THEOREM 2. *If X is a proper convex region in the complex plane, then*

$$|\nabla \log \lambda_X(z)| \le \lambda_X(z)$$

for all z in X.

Proof. Fix $a \in X$ and let $z = f(w)$ be a holomorphic universal covering projection of $(D, 0)$ onto (X, a). Then

$$\lambda_X(f(w))|f'(w)| = \frac{2}{1 - |w|^2}.$$

In particular, $\lambda_X(a) = 2/|f'(0)|$. Also

$$\log \lambda_X(f(w)) + \frac{1}{2}\log f'(w) + \frac{1}{2}\log \overline{f'(w)} = \log 2 - \log(1 - w\overline{w}).$$

We apply the operator $\partial/\partial w$ to both sides of this identity and obtain

$$\frac{\partial \log \lambda_X(f(w))}{\partial z} f'(w) + \frac{1}{2}\frac{f''(w)}{f'(w)} = \frac{\overline{w}}{1-w\overline{w}}.$$

For $w = 0$ this gives

$$\frac{\partial \log \lambda_X(a)}{\partial z} f'(0) = -\frac{1}{2}\frac{f''(0)}{f'(0)},$$

so that

$$(3) \qquad \begin{aligned} |\nabla \log \lambda_X(a)| &= 2\left|\frac{\partial \log \lambda_X(a)}{\partial z}\right| \\ &= \frac{1}{|f'(0)|}\frac{|f''(0)|}{|f'(0)|} \\ &= \frac{1}{2}\frac{|f''(0)|}{|f'(0)|}\lambda_X(a). \end{aligned}$$

Since X is convex, it follows that the function

$$\frac{f(z)-f(0)}{f'(0)} = z + a_2 z^2 + \cdots$$

is a normalized convex univalent function in D, so $|a_2| \leq 1$, or $|f''(0)/f'(0)| \leq 2$. This inequality in conjunction with (3) yields the desired result.

3. Euclidean and hyperbolic curvature

In this section we prove the converse of Theorem 2 by making use of euclidean and hyperbolic curvature. We begin with a discussion of euclidean curvature. Let $\gamma : z = z(t),\ t \in [a,b]$, be a C^2 curve in the complex plane with $z'(t) \neq 0$ for $t \in [a,b]$. The euclidean curvature $K_e(z,\gamma)$ of the curve γ at the point $z = z(t)$ is the rate of change of the angle θ that the tangent vector makes with the positive real axis with respect to arc length:

$$\begin{aligned} K_e(z,\gamma) &= \frac{d\theta}{ds} = \frac{d\theta}{dt}\frac{dt}{ds} \\ &= \frac{1}{|z'(t)|}\operatorname{Im}\{\frac{z''(t)}{z'(t)}\}. \end{aligned}$$

The value of the curvature is independent of the parametrization of γ. This is the signed curvature; its value is negated if the path is traversed in the oppositve direction. If f is holomorphic and locally univalent in a neighborhood of γ, then $f \circ \gamma$ is also a C^2 curve with nonvanishing tangent. The formula for the change of euclidean curvature under f is given by [3]

$$K_e(f(z), f \circ \gamma)|f'(z)| = K_e(z, \gamma) + \mathrm{Im}\{\frac{f''(z)}{f'(z)} \frac{z'(t)}{|z'(t)|}\}.$$

Next, we consider hyperbolic curvature. A discussion of hyperbolic curvature is given in [2] and [10]. Suppose X is a hyperbolic region in the complex plane and γ is a C^2 curve in X with nonvanishing tangent. The hyperbolic curvature of γ at $z = z(t)$ is given by

$$\begin{aligned} K_X(z, \gamma) &= \frac{1}{\lambda_X(z)} \left[K_e(z, \gamma) + 2\mathrm{Im}\{\frac{\partial \log \lambda_X(z)}{\partial z} \frac{z'(t)}{|z'(t)|}\}\right] \\ &= \frac{1}{\lambda_X(z)} \left[K_e(z, \gamma) - \frac{\partial \log \lambda_X(z)}{\partial n}\right], \end{aligned}$$

where $n = n(z)$ is the unit normal to γ at z.

Example. Let us determine the hyperbolic curvature of the positively oriented circle γ in D with center 0 and radius $r \in (0,1)$. We note that

$$\begin{aligned} K_D(z, \gamma) &= \frac{1}{\lambda_D(z)} \left[K_e(z, \gamma) + 2\mathrm{Im}\{\frac{\overline{z}}{1 - |z|^2} \frac{z'(t)}{|z'(t)|}\}\right] \\ &= \frac{1 - |z|^2}{2} K_e(z, \gamma) + \mathrm{Im}\{\frac{\overline{z} z'(t)}{|z'(t)|}\}. \end{aligned}$$

A parametrization of γ is $z = z(t) = re^{it}$, $0 \le t \le 2\pi$. Then

$$K_e(z, \gamma) = \frac{1}{|z'(t)|} \mathrm{Im}\{\frac{z''(t)}{z'(t)}\} = \frac{1}{r}.$$

Also $\overline{z(t)} z'(t)/|z'(t)| = ir$ so that

$$K_D(z, \gamma) = \frac{1 - r^2}{2} \frac{1}{r} + r = \frac{1}{2}(r + \frac{1}{r}).$$

Note that $r + \frac{1}{r} > 2$, so any circle in D with center 0 has hyperbolic curvature strictly larger than 1.

Now, we show that hyperbolic curvature is invariant under conformal mappings.

LEMMA 3. *Suppose X and Y are hyperbolic regions and $f : X \to Y$ is a conformal mapping of X onto Y. Then $K_X(z,\gamma) = K_Y(f(z), f \circ \gamma)$ for any path γ in X.*

Proof. Let $w = f(z)$ and $\delta = f \circ \gamma$. The transformation law for euclidean curvature gives

$$K_e(z,\gamma) = K_e(w,\delta)|f'(z) - \mathrm{Im}\frac{f''(z)}{f'(z)}\frac{z'(t)}{|z'(t)|}. \tag{4}$$

From $\lambda_X(z) = \lambda_Y(f(z))|f'(z)|$, we obtain

$$\log \lambda_X(z) = \log \lambda_Y(f(z)) + \frac{1}{2}\log f'(z) + \frac{1}{2}\log \overline{f'(z)}.$$

Then

$$\frac{\partial \log \lambda_X(z)}{\partial z} = \frac{\partial \log \lambda_Y(f(z))}{\partial w} f'(z) + \frac{1}{2}\frac{f''(z)}{f'(z)}$$

so that

$$2\mathrm{Im}\left\{\frac{\partial \log \lambda_X(z)}{\partial z}\frac{z'(t)}{|z'(t)|}\right\} = 2\mathrm{Im}\left\{\frac{\partial \log \lambda_Y(w)}{\partial w} f'(z)\frac{z'(t)}{|z'(t)|}\right\} + \mathrm{Im}\left\{\frac{f''(z)}{f'(z)}\frac{z'(t)}{|z'(t)|}\right\}.$$

Next, $w'(t) = f'(z)z'(t)$ gives $|w'(t)| = |f'(z)||z'(t)|$ and

$$f'(z)\frac{z'(t)}{|z'(t)|} = |f'(z)|\frac{w'(t)}{|w'(t)|}.$$

Hence

$$(5)\quad 2\mathrm{Im}\left\{\frac{\partial \log \lambda_X(z)}{\partial z}\frac{z'(t)}{|z'(t)|}\right\}$$
$$= 2|f'(z)|\mathrm{Im}\left\{\frac{\partial \log \lambda_Y(w)}{\partial w}\frac{w'(t)}{|w'(t)|}\right\} + \mathrm{Im}\left\{\frac{f''(z)}{f'(z)}\frac{z'(t)}{|z'(t)|}\right\}.$$

From (4) and (5), we obtain

$$\begin{aligned}
&K_X(z,\gamma)\\
=&\frac{1}{\lambda_X(z)}\left[K_e(z,\gamma)+2\mathrm{Im}\{\frac{\partial \log \lambda_X(z)}{\partial z}\frac{z'(t)}{|z'(t)|}\}\right]\\
=&\frac{1}{\lambda_Y(w)|f'(z)|}\left[K_e(w,\delta)|f'(z)|+2|f'(z)|\mathrm{Im}\left\{\frac{\partial \log \lambda_Y(w)}{\partial w}\frac{w'(t)}{|w'(t)|}\right\}\right]\\
=&K_Y(f(z),\ f\circ\gamma).
\end{aligned}$$

Finally, we establish the converse of Theorem 2. It is well known [6, p.29] that if X is a region in the complex plane bounded by a simple closed curve, ∂X, of class C^2, then X is convex if and only if the euclidean curvature of ∂X, with respect to a fixed orientation, is always nonnegative.

THEOREM 4. *If X is a hyperbolic region in the complex plane such that $|\nabla \log \lambda_X(z)| \le \lambda_X(z)$ for all z in X, then X is convex.*

Proof. Let f be a holomorphic universal covering projection of D onto X, and let γ be the circle $w = w(t) = re^{it}$, $0 \le t \le 2\pi$, $0 < r < 1$. Then Lemma 3 implies that

$$K_X(f(w),\ f\circ\gamma) = K_D(w,\gamma).$$

The previous Example gives $K_D(w,\gamma) > 1$. Let $z = f(w)$. Then, by hypothesis, we have

$$\begin{aligned}
K_e(z,f\circ\gamma) &= K_X(z,f\circ\gamma)\lambda_X(z) - 2\mathrm{Im}\{\frac{\partial \log \lambda_X(z)}{\partial z}\frac{z'(t)}{|z'(t)|}\}\\
&\ge K_X(z,f\circ\gamma)\gamma_X(z) - 2\left|\frac{\partial \log \lambda_X(z)}{\partial z}\right|\\
&> \lambda_X(z) - |\nabla \log \lambda_X(z)|\\
&\ge 0.
\end{aligned}$$

Thus, $f(\{w : |w| < r\})$ is convex. This holds for all $r \in (0,1)$, so $f(D) = X$ is convex.

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