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NOTES ON THE FRESNEL INTEGRABLE FUNCTIONS *

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I. Introductory Preliminaries

Let H be a separable Hilbert space over $\mathbf{R}$. Let $M(H)$ be the collection of $\mathbf{C}$-valued, countably additive measures on $\mathcal{B}(H)$, the Borel class of H. $M(H)$ is a Banach algebra under the total variation norm where the convolution is taken as the multiplication. Given μ in $M(H)$, $\hat{\mu}$ is defined for every r in H by the formula

$$\hat{\mu}(r) = \int_H \exp\{i(r,h)\}d\mu(h). \tag{1.1}$$

Let $\mathcal{F}(H) = \{\hat{\mu} \mid \mu \in M(H)\}$. The correspondence $\mu \to \hat{\mu}$ is injective and carries convolution into pointwise multiplication. Hence, letting $\|\hat{\mu}\| = \|\mu\|$, we have that $\mathcal{F}(H)$ is a Banach algebra. The Fresnel integral $\mathcal{F}(\hat{\mu})$ is defined for $\hat{\mu}$ in $\mathcal{F}(H)$ by the formula

$$\mathcal{F}(\hat{\mu}) = \int_H \exp\{-\frac{i}{2}\|h\|^2\}d\mu(h). \tag{1.2}$$

The space $\mathcal{F}(H)$ plays a key role throughout the fundamental monograph [2] of Albeverio and Høegh-Krohn.

Fix $t > 0$. Let H_t be the space of $\mathbf{R}$-valued functions r on $[0,t]$ which are absolutely continuous with square integrable derivative Dr and which satisfy $r(t) = 0$. H_t is a separable Hilbert space over $\mathbf{R}$ with inner product

$$(r_1,r_2) = \int_0^t (Dr_1)(s)(Dr_2)(s)ds. \tag{1.3}$$

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Functions on H_t of the form

$$g(r) = \psi(r(0)), \tag{1.4}$$

where $\psi : \mathbf{R} \to \mathbf{C}$, are simple but crucial for the applications of the theory to quantum mechanics. A very simple result of Albeverio and Høegh–Krohn [2] shows that if $\psi = \hat{\nu}$ where ν is in $M(\mathbf{R})$, then g is in $\mathcal{F}(H_t)$, the Fresnel class of H_t. Because of the great usefulness of the Fresnel class, it is natural to ask if g of the form (1.4) is in $\mathcal{F}(H_t)$ for some class of $\psi' s$. In their paper [6], Chang, Johnson and Skoug showed that the answer is "No" ; that is, if g is in $\mathcal{F}(H_t)$, then there exists ν in $M(\mathbf{R})$ such that $\psi = \hat{\nu}$. And also Johnson proved in his paper [7] that $\mathcal{F}(H_t)$ is equivalent to the space S which is a Banach algebra of analytic Feynman integrable functionals.

There is a particular Hilbert space H_Q which is an extension of Albeverio and Høegh–Krohn's H_t. H_Q is the space in which we will be concerned throughout this paper.

Fix $p, q > 0$ and let $Q = [0,p]\times[0,q]$. Let H_Q be the set of all functions $r : Q \to \mathbf{R}$ for which there exists v in $L_2(Q)$ such that for all (s,t) in Q

$$r(s,t) = \int_s^p \int_t^q v(\tau_1,\tau_2)d\tau_1 d\tau_2. \tag{1.5}$$

The inner product on H_Q is defined by

$$(r_1,r_2)_{H_Q} = \int_0^p \int_0^q (\frac{\partial^2 r_1}{\partial s\partial t})(s,t)(\frac{\partial^2 r_2}{\partial s\partial t})(s,t)dsdt. \tag{1.6}$$

H_Q, equipped with this inner product, is a separable infinite dimensional Hilbert space over $\mathbf{R}$. It will be helpful to introduce the family of functions $\{r_{\tau_1,\tau_2} : (\tau_1,\tau_2) \in Q\}$ from H_Q ;

$$r_{\tau_1,\tau_2}(s,t) = \min\{p-s, p-\tau_1\}\min\{q-t, q-\tau_2\}. \tag{1.7}$$

These functions have the reproducing property,

$$(r, r_{\tau_1,\tau_2})_{H_Q} = r(\tau_1,\tau_2) \quad \text{for all} \quad r \quad \text{in} \quad H_Q,$$

and also H_Q is the reproducing kernel Hilbert space associated with two parameter Brownian motion.

In this paper, we show that various functions belong to $\mathcal{F}(H_Q)$, the space of Fresnel integrable functions on H_Q. And we establish necessary and sufficient conditions for the Fresnel integrability of certain class of functions on H_Q.

II. Some Fresnel Integrable Functions on H_Q

In their paper [5], Chang, Johnson and Skoug established a main theorem. After the statement of this theorem, we find various Fresnel integrable functions on H_Q as its corollaries.

THEOREM. *(1.) Let H be a separable infinite dimensional Hilbert space over $\mathbf{R}$.*

(2.) Let $(Y, \mathcal{Y}, \eta)$ be a measure space where η is either a non–negative, σ–finite measure or a $\mathbf{C}$–valued measure.

(3.) Let $\emptyset_{i,j} : Y \to H$ be $\mathcal{Y} - \mathcal{B}(H)$ measurable for $i = 1, \ldots, l$, $j = 1, \ldots, m$.

(4.) Let $\theta : Y \times \mathbf{R}^{lm} \to \mathbf{C}$ be given by $\theta(y; \cdot) = \hat{\nu}_y(\cdot)$ where ν_y is in $M(\mathbf{R}^{lm})$ for every y in Y and where the family $\{\nu_y : y \in Y\}$ satisfies : (i) $\nu_y(B)$ is a $\mathcal{Y}$–measurable function of y for every B in $\mathcal{B}(\mathbf{R}^{lm})$, and (ii) $\|\nu_y\|$ is in $L_1(Y, \mathcal{Y}, |\eta|)$.

Under these hypotheses, $f : H \to \mathbf{C}$ given by

$$f(r) = \int_Y \theta(y : <r, \emptyset_{1,1}(y)), \ldots, (r, \emptyset_{l,m}(y))>) d\eta(y) \tag{2.1}$$

belong to $\mathcal{F}(H)$ and satisfies the inequality

$$\|f\| \le \int_Y \|\nu_y\| d|\eta|(y). \tag{2.2}$$

Further, since $\mathcal{F}(H)$ is a Banach algebra, g is in $\mathcal{F}(H)$ where

$$g(r) = \exp\{f(r)\}. \tag{2.3}$$

REMARKS. *(1) It suffices to assume in (4.) that $\theta(y, \cdot) = \hat{\nu}_y(\cdot)$ for η–a.e. y in Y.*

(2) Since $\mathcal{F}(H)$ is a Banach algebra, many analytic functions of f can formed. We explicitly mention the exponential function in (2.3) because it plays a central role in the quantum theory.

Our first two corollaries are the extension of simple results of Albeverio and Høegh–Krohn's [2].

COROLLARY 1. *Let $\psi = \hat{\nu}$ where ν is in $M(\mathbf{R})$. Define $f_1 : H_Q \to C$ by*

$$f_1(r) = \psi(r(0,0)). \tag{2.4}$$

Then f_1 belongs to $\mathcal{F}(H_Q)$.

Proof. Apply the above theorem after making the following choices ; $H = H_Q, (Y, \mathcal{Y}, \eta) = (Q, \mathcal{B}(Q), \eta)$ where η is any probability measure, $l = m = 1$ and $\emptyset_{1,1}(s,t) = r_{0,0}(s,t)$ as in (1.7), $\theta((s,t),\cdot) = \hat{\nu}(\cdot)$. With these choices, the right hand side of (2.1) becomes

$$\int_Q \hat{\nu}((r, r_{0,0})_{H_Q}) d\eta(s,t) = \psi(r(0,0)),$$

and the result follows.

COROLLARY 2. *Let $\theta = \hat{\nu}$ where ν is in $M(\mathbf{R})$. Define $f_2 : H_Q \to \mathbf{C}$ by*

$$f_2(r) = \int_Q \theta(r(s,t)) ds\, dt. \tag{2.5}$$

Then f_2 belongs to $\mathcal{F}(H_Q)$.

Proof. Take $H, Y, \eta, l, m,$ and θ as in the proof of Corollary 1. Let η be Lebesgue measure on Q and take $\emptyset_{1,1}(s,t) = r_{s,t}$ as in (1.7). Then the right hand side of (2.1) is just

$$\int_Q \theta(r, r_{s,t})_{H_Q}) ds\, dt = \int_Q \theta(r(s,t)) ds\, dt,$$

and the result follows.

Let $H_Q^{lm} = \overset{lm}{\underset{1}{\times}} H_Q$ consist of functions $r : Q \to \mathbf{R}^{lm}$ such that each component $r_{i,j}$ is in H_Q. Define the inner product of r and r^* in H_Q^{lm} as the sum of the H_Q inner products of the components.

COROLLARY 3. *Let $\theta = \hat{\nu}$ where ν is in $M(\mathbf{R}^{lm})$. Define $f_3 : H_Q^{lm} \to \mathbf{C}$ by*

$$f_3(r) = \int_Q \theta(r_{1,1}(s,t), \ldots, r_{l,m}(s,t)) ds\, dt. \tag{2.6}$$

Then f_3 belongs to $\mathcal{F}(H_Q^{lm})$.

Proof. Apply the above theorem after making the following choices ; $H = H_Q^{lm}$, $(Y, \mathcal{Y}, \eta)$ as in Corollary 2, $\theta((s,t), \cdot) = \hat{\nu}(\cdot)$, $\emptyset_{i,j}(s,t)$ the function in H_Q^{lm} which is 0 except in the $(i,j)^{th}$ component where it is $r_{s,t}$.

The next corollary is an extension of Corollary 4 in [5].

COROLLARY 4. *Let $\theta : Q \times \mathbf{R} \to \mathbf{C}$ be given by $\theta((s,t), \cdot) = \hat{\nu}_{s,t}(\cdot)$ where $\nu_{s,t}$ is in $M(\mathbf{R})$ for every $(s,t) \in Q$ and where the family $\{\nu_{s,t} : (s,t) \in G\}$ satisfies : (i) $\nu_{s,t}(B)$ is a Borel measurable function of (s,t) for every B in $\mathcal{B}(\mathbf{R})$, and (ii) $\|\nu_{s,t}\|$ is integrable over Q with respect to Lebesgue measure. Define $f_4 : H_Q \to \mathbf{C}$ by*

$$f_4(r) = \int_Q \theta((s,t), r(s,t)) ds\, dt. \tag{2.7}$$

Then f_4 belongs to $\mathcal{F}(H_Q)$.

Proof. Make the choice of H, Y, etc. from the above theorem as in Corollary 2 except taking $\theta((s,t), \cdot) = \hat{\nu}_{s,t}(\cdot)$.

The following is as in Corollary 4 except that Lebesgue measure is replaced by a general Borel measure η on Q.

COROLLARY 5. *Let $\theta : Q \times \mathbf{R} \to \mathbf{C}$ be given by $\theta((s,t), \cdot) = \hat{\nu}_{s,t}(\cdot)$ where $\nu_{s,t}$ is in $M(\mathbf{R})$ for every (s,t) in Q and where the family $\{\nu_{s,t} : (s,t) \in Q\}$ satisfies : (i) $\nu_{s,t}(B)$ is a Borel measurable function of (s,t)*

for every B in $\mathcal{B}(R)$, and (ii) $\|\nu_{s,t}\|$ is integrable over Q with respecto to $|\eta|$ where η is a Borel measure on Q. Define $f_5 : H_Q \to \mathbf{C}$ by

$$f_5(r) = \int_Q \theta((s,t), r(s,t))d\eta(s,t). \tag{2.8}$$

Then f_5 belongs to $\mathcal{F}(H_Q)$.

Proof. Apply the above theorem with $H = H_Q$, $Y = Q$, $\mathcal{Y} = \mathcal{B}(Q)$, $l = m = 1$ and $\emptyset_{1,1}(s,t) = r_{s,t}$ where $r_{s,t}$ is given by (1.7).

III. Necessary and Sufficient Conditions for the Fresnel Integrability on H_Q

In this section, we establish necessary and sufficient conditions for the Fresnel integrability of certain class of functions on H_Q which are similar to those on H_t.

THEOREM 1. *Let $0 \le s_1 < s_2 < \cdots < s_l < p$, $0 \le t_1 < t_2 < \cdots < t_m < q$, and let ν be in $M(\mathbf{R}^{lm})$. Define $f : H_Q \to \mathbf{C}$ by*

$$f(r) = \hat{\nu}(r(s_1,t_1), \ldots, r(s_l,t_m)). \tag{3.1}$$

Then f is in $\mathcal{F}(H_Q)$; in fact, there exists a unique measure μ in $M(H_Q)$ such that

$$\hat{\mu}(r) = \hat{\nu}(r(s_1,t_1), \ldots, r(s_l,t_m)) \tag{3.2}$$

for all r in H_Q.

Proof. Let $\emptyset : \mathbf{R}^{lm} \to H_Q$ be defined by

$$\emptyset(a_{1,1}, \ldots, a_{l,m}) = a_{1,1}r_{1,1} + \cdots + a_{l,m}r_{l,m}$$

where $r_{i,j} \equiv r_{s_i,t_j}$ as in (1.7) for $i = 1, \ldots, l$, $j = 1, \ldots, m$. Let $\mu = \nu \circ \emptyset^{-1}$. Then μ is in $M(H_Q)$. By the linear independence of $r_{1,1}, \ldots, r_{l,m}$

and the change of variable formula, we can write, for any r in H_Q,

$$\begin{aligned}
&\hat{\mu}(r)\\
&=\int_{H_Q} \exp\{i(r,h)\}d\mu(h)\\
&=\int_{H_Q} \exp\{i(r,h)\}d(\nu \circ \emptyset^{-1})(h)\\
&=\int_{\mathbf{R}^{lm}} \exp\{i(r,\emptyset(a_{1,1},\ldots,a_{l,m}))\}d\nu(a_{1,1},\ldots,a_{l,m})\\
&=\int_{\mathbf{R}^{lm}} \exp\{i(r,a_{1,1}r_{1,1}+\cdots+a_{l,m}r_{l,m})\}d\nu(a_{1,1},\ldots,a_{l,m})\\
&=\int_{\mathbf{R}^{lm}} e^{\{i(((r,r_{1,1}),\ldots,(r,r_{l,m})),(a_{1,1},\ldots,a_{l,m}))\}}d\nu(a_{1,1},\ldots,a_{l,m})\\
&=\hat{\nu}(r(s_1,t_1),\ldots,r(s_l,t_m)).
\end{aligned}$$

Finally the uniqueness of a measure μ satisfying (3.2) is a consequence of the fact that the map $\mu \to \hat{\mu}$ is one–one.

Note that Corollary 1 in Section 2 is just the special case of Theorem 1 with $l = m = 1$ and $s_1 = t_1 = 0$.

THEOREM 2. *Let* $0 \le s_1 < s_2 < \cdots < s_l < p$, $0 \le t_1 < t_2 < \cdots < t_m < q$, *and let* $\psi : \mathbf{R}^{lm} \to \mathbf{C}$. *Suppose that there exists* μ *in* $M(H_Q)$ *such that for all* r *in* H_Q.

$$\hat{\mu}(r) = \psi(r(s_1,t_1),\cdots,r(s_l,t_m)).$$

Then there exists a measure ν *in* $M(\mathbf{R}^{lm})$ *such that* $\psi = \hat{\nu}$ *on* $\mathbf{R}^{lm}$.

Proof. Let $r_{i,j} \equiv r_{s_i,t_j}$ as in (1.7) for $i = 1,\ldots,l$, $j = 1,\ldots,m$, and let $[r_{1,1},\ldots,r_{l,m}]$ be the span of $r_{1,1},\ldots,r_{l,m}$. By the linear independence of $r_{1,1},\ldots,r_{l,m}$, we know $\dim[r_{1,1},\ldots,r_{l,m}] = lm$, and hence $\{((r,r_{1,1}),\ldots,(r,r_{l,m})) : r \in [r_{1,1},\ldots,r_{l,m}]\} = \mathbf{R}^{lm}$.

By the Gram–Schmidt process, we get an orthonormal set $\{e_{1,1},\ldots, e_{l,m}\}$ which is a basis for $[r_{1,1},\ldots,r_{l,m}]$. For each $i = 1,\ldots,l$, $j = 1,\ldots,m$,

$$r_{i,j} = (r_{i,j},e_{1,1})e_{1,1} + \cdots + (r_{i,j},e_{l,m})e_{l,m}.$$

Hence

$$\begin{aligned}\hat{\mu}(r) &= \psi((r, r_{1,1}), \dots, (r, r_{l,m})) \\ &= ((r_{1,1}, e_{1,1})(r, e_{1,1}) + \cdots + (r_{1,1}, e_{l,m})(r, e_{l,m}), \\ &\quad \dots\dots\dots\dots, \\ &\quad (r_{l,m}, e_{1,1})(r, e_{1,1}) + \cdots + (r_{l,m}, e_{l,m})(r, e_{l,m})) \\ &= B((r, e_{1,1}), \dots, (r, e_{l,m}))\end{aligned}$$

where B is the linear map from $\mathbf{R}^{lm}$ onto $\mathbf{R}^{lm}$ sending $((r, e_{1,1}), \dots, (r, e_{l,m}))$ to $((r, r_{1,1}), \dots, (r, r_{l,m}))$.

By Proposition 6 in [6], there exists η in $M(\mathbf{R}^{lm})$ such that $\hat{\eta} = \psi \circ B$ on $\mathbf{R}^{lm}$. Applying Lemma 7 in [6] with $T = B^{-1}$, we see that $\psi = (\psi \circ B) \circ B^{-1}$ is the Fourier transform of some measure $\nu = \eta \circ B^t$ in $M(\mathbf{R}^{lm})$, that is, $\psi = \hat{\nu}$ for some ν in $M(\mathbf{R}^{lm})$.

References

1. J.M. Ahn, K.S. Chang and I. Yoo, *Some Banach algebras of Yeh–Feynman Integrable Functions,* J. Korean Math. Soc. Vol. 24(1987), 257–266.
2. S. Albeverio and R. Høegh–Krohn, *Mathematical Theory of Feynman Path Integrals*, Lecture Notes in Math. 523, Springer–Verlag, 1976.
3. E. Berkson and T.A. Gillespie, *Absolutely Continuous Functions of Two Variables and Well–Bounded Operators*, J. London Math. Soc. 30(1984), 305–321.
4. R.H. Cameron and D.A. Storvick, *Some Banach Algebras of Analytic Feynman Integrable Functionals*, Lecture Notes in Math. 798, Springer–Verlag, 1980.
5. K.S. Chang, G.W. Johnson and D.L. Skoug, *Functions in the Fresnel Class*, Proc. Amer. Math. Soc. Vol. 100, No.2(1987), 309–318.
6. K.S. Chang, *Necessary and Sufficient Conditions for the Fresnel Integrability of Certain Classes of Functions*, J. Korean Math. Soc. 21(1984), 21–29.
7. G.W. Johnson, *The Equivalence of Two Approaches to the Feynman Integral*, J. Math. Phys. 23(1982), 2090–2096.
8. G. Kallianpur, *Abstract Wiener Processes and Their Reproducing Kernel Hilbert Space*, Z. Wahrscheinlichkeitstheorie verw. Gerb. 17(1971), 113–123.
9. G. Kallianpur and C. Bromley, *Generalized Feynman Integrals using Analytic Continuation in Several Complex Variables*, Stochastic Analysis and Applications (M.A. Pinsky, editor), Dekker, New York, 1984.

10. C. Park, *A Generalized Paley–Wiener–Zygmund Integral and its Applications*, Proc. Amer. Math. Soc. 23(1969), 388–400.

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