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DOMAINS OF $\mathcal{C}$–HOLOMORPHY IN BANACH SPACES

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1. Introduction

Let E be a separable Banach space and U be an open subset of E. It is well known that every domain of existence in E is a domain of holomorphy. But the converse is not solved yet. But Dineen [2] and Matos [4] succeeded in formulating the Cartan–Thullen theorem for a separable Banach space by using $H_b(U)$ and $H_\tau(U)$, respectively. For nonseparable infinite dimensional Banach spaces it is known that the above converse does not hold and a counterexample was given by Hirschowitz [3].

In this article, using $H_I(U)$ we are going to introduce the concept of a domain of $\mathcal{C}$–holomorphy, which is stronger than that of a domain of holomorphy but weaker than that of a domain of existence in general, and next show that the equivalence between a domain of existence and a domain of $\mathcal{C}$–holomorphy in a separable Banach space, i.e., we will show the followings.

(a) U is a domain of existence.
(b) U is a domain of $\mathcal{C}$–holomorphy.
(c) U is a domain of holomorphy.

2. Domains of $\mathcal{C}$–holomorphy

Let E be a Banach space, U be a nonempty open subset of E and ∂U be the set of all boundary points of U. $H(U)$ denotes the vector space of all complex valued holomorphic functions on U. Let I be a countable cover of U by nonempty open subsets of U. We denote by $H_I(U)$ the vector space of all complex valued holomorphic functions on U which

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are bounded on each open subset of I. The natural topology of $H_I(U)$ is the Hausdorff locally convex topology defined by the seminorms

$$P_V : f \in H_I(U) \mapsto P_V(f) = \sup_{x \in V} |f(x)|$$

where V ranges over I.

Then it is well known that $H_I(U)$ with the natural topology is a Fréchet space. For details, see [1]. Let $\mathcal{C}$ be the set of all countable covers of U by open subsets, then $H(U) = \bigcup_{I \in \mathcal{C}} H_I(U)$.

DEFINITION 1. *A nonempty connected open subset U of E is said to be a domain of $H_I(U)$–holomorphy if there does not exist a pair of nonempty connected open sets V and W in E such that*

(a) $W \subset U \cap V$ and V is not contained in U.

(b) For every $f \in H_I(U)$ there exists $g \in H(V)$ such that $g|_W = f|_W$.

DEFINITION 2. *Let U be a nonempty connected open subset of E. Let $f \in H_I(U)$ and $\xi \in \partial U$. f is said to be H_I–regular at ξ if there exists a pair of nonempty connected open sets V, W such that $W \subset U \cap V$, $\xi \in V$ (which implies that $V \not\subset U$), and there exists $g \in H(V)$ such that $g|_W = f|_W$. Conversely, ξ is said to be a H_I–singular point for f if no such pair of sets exist. f is said to be H_I–singular on ∂U if every point of ∂U is a H_I–singular point of f. This means that for all nonempty connected open subsets V, W of E with $W \subset U \cap V$ and $V \not\subset U$, there is no $g \in H(V)$ for which $g = f$ in W. $S_I(U)$ will denote the set of all $f \in H_I(U)$ which are H_I–singular at every point of ∂U. U is said to be a H_I–domain of existence if $S_I(U) \neq \phi$.*

DEFINITION 3. *A nonempty connected open subset U of E is said to be a domain of $\mathcal{C}$–holomorphy if there exists a countable open cover $I = (V_n)$ of U such that U is a domain of $H_I(U)$–holomorphy.*

THEOREM 4. *Suppose taht Eis a separable Banach space, and let U be a nonempty connected open subset of E. Then the following are equivalent:*

(a) U is a domain of $H_I(U)$–holomorphy.

(b) U is a $H_I(U)$–domain of existence.

(c) The complement $\mathbf{C}S_I(U)$ of $S_I(U)$ in $H_I(U)$ is of first category in $H_I(U)$.

Proof. We prove (c) $\Longrightarrow$ (b) first. If $\mathbf{C}S_I(U)$ is of first category in $H_I(U)$ then, $S_I(U) \neq \phi$. In fact, if $S_I(U) = \phi$, then $\mathbf{C}S_I(U) = H_I(U)$ will be of first category, which is a contradiction of Baire's theorem, since $H_I(U)$ is a complete metric space in the natural topology.

(b) $\Longrightarrow$ (a) is obvious. Next we show (a) $\Longrightarrow$ (c). Let V and W be nonempty connected open subsets of E such that $W \subset U \cap V$ and $V \not\subset U$. $H_I(U, V, W)$ denotes the subalgebra of $H_I(U)$ consisting of all functions $f \in H_I(U)$ for which there exists (necessarily unique) $g \in H(V)$ such that $f = g$ in W. For each $m \in \mathbf{N}$, let $H_{I,m}(U, V, W)$ be the convex subset of $H_I(U, V, W)$ consisting of all $f \in H_I(U, V, W)$ for which the corresponding $g \in H(V)$ satisfies the relation $|g| \leq m$ in V.

Then we can show that $H_{I,m}(U, V, W)$ is closed and nowhere dense in $H_I(U)$, and $\mathbf{C}S_I(U)$ is the union of countable family of nowhere dense sets of the form $H_{I,m}(U, V, W)$. The detailed proof of them is similiar to that of Theorem 2 in Matos [4]. Therefore $\mathbf{C}S_I(U)$ is of first category in $H_I(U)$.

THEOREM 5. *Suppose that E is a separable Banach space and let U be a nonempty connected open subset of E. Then the following are equivalent:*

(a) U is a domain of existence.

(b) U is a domain of $\mathcal{C}$–holomorphy.

Proof. Suppose that U is a domain of existence. Then there exists $f \in H(U)$ such that U is the domain of existence of the function f. Let $V_n = \{x \in U; |f(x)| < n\}$. Then $I = (V_n)$ is an open countable cover of U such that U is a $H_I(U)$–domain of existence. Conversely, if there exists an open countable cover $I = (V_n)$ of U such that U is a $H_I(U)$–domain of existence then it is clear that U is a domain of existence. Hence we showed that U is a domain of existence if and only if there exists an open countable cover $I = (V_n)$ of U such that U is a $H_I(U)$–domain of existence. By Theorem 4, U is a domain of existence if and only if there exists an open countable cover $I = (V_n)$ of U such that U is a domain of $H_I(U)$-holomorphy. Hence U is a domain of existence if and only if U is a domain of $\mathcal{C}$–holomorphy from definition.

REMARK 6. *Every domain of* $\mathcal{C}$*–holomorphy is a domain of holomorphy in Banach space. But we don't know if every domain of holomorphy is a domain of* $\mathcal{C}$*–holomorphy in a separable Banach space.*

COROLLARY 7. *Suppose that E is a separable Banach space with the bounded approximation property, and let U be a nonempty connected subset of E. Then the following are equivalent:*
(a) U is a domain of existence.
(b) U is a domain of $\mathcal{C}$*–holomorphy.*
(c) U is a domain of holomorphy.
(d) U is holomorphically convex.
(e) U is pseudoconvex.

Proof. This follows immediately from Theorem 5 and Theorem 45.8 in [5].

References

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