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THE HYPOELLIPTICITY OF CONVOLUTION EQUATIONS IN THE GENERALIZED DISTRIBUTION SPACES OF BEURLING TYPE

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1. Introduction

In [4] Ehrenpreis studied the following equation which is called the hypoelliptic convolution equation in the distribution space $\mathcal{D}'(\mathbf{R}^n)$:

$$S * u = v, \ u \in \mathcal{D}'(\mathbf{R}^n)$$

implies $u \in \mathcal{E}(\mathbf{R}^n)$ whenever v does, where the convolution operator S is given in $\mathcal{E}'(\mathbf{R}^n)$. He have shown that S is hypoelliptic in $\mathcal{D}'$ iff there exist constants B and M such that

$$|\hat{S}(\xi)| \geq |\xi|^{-B} \text{ for } |\xi| \geq M \ \xi \in \mathbf{R}^n$$

and

$$|\mathrm{Im}\ \zeta|/\log|\zeta| \to \infty \text{ if } |\zeta| \to \infty \text{ in } C^n \text{ and } \hat{S}(\zeta) = 0.$$

Later Chou [3] studied the same problem in the ultradistribution space of Roumieu type.

In this paper we have studied the same proble in the generalized distribution spaces of Beurling type. For this we briefly review the generalized distribution spaces and their properties which we need in this paper. Further details are given in [2]. We denote by M_c the set of all continuous real valued functions ω on $\mathbf{R}^n$ satisfying the following conditions:

$$(\alpha) \qquad 0 = \omega(0) \leq \omega(\xi + \eta) \leq \omega(\xi) + \omega(\eta), \ \xi, \eta \in \mathbf{R}^n$$

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$$(\beta) \qquad \int_{\mathbf{R}^n} \frac{\omega(\xi)}{(1+|\xi|)^{n+1}} d\xi < \infty$$

(γ) $\omega(\xi) \geq a + b \log(1+|\xi|)$ for some constant a and $b > 0$.

(δ) $\omega(\xi) = \sigma(|\xi|)$ for an increasing concave function σ on $[0, \infty)$.

For example, $\omega(\xi) = \log(1+|\xi|)$ and $\omega(\xi) = |\xi|^{\frac{1}{d}}$, $d > 1$ satisfy all the conditions. Throughout this paper ω represents an element in M_c and Ω is an open set in $\mathbf{R}^n$.

Let $\mathcal{D}_\omega(\Omega)$ be the set of all ϕ in $L^1(\mathbf{R}^n)$ such that ϕ has compact support in Ω and

$$\|\phi\|_\lambda = \int_{\mathbf{R}^n} |\hat{\phi}(\xi)| e^{\lambda\omega(\xi)} d\xi < \infty \text{ for every } \lambda > 0.$$

The topology on this space is given by the inductive limit topology of Frechet space $\mathcal{D}_\omega(K)$ induced by the above semi–norms where K is a compact set in Ω. We denote by $\mathcal{E}_\omega(\Omega)$ the set of all complex valued functions in Ω such that $\phi\psi$ is in $\mathcal{D}_\omega(\Omega)$ for every $\phi \in \mathcal{D}_\omega(\Omega)$, equipped the topology generated by semi–norms $\|\phi\psi\|_\lambda$ for every $\phi \in \mathcal{D}_\omega(\Omega)$ and $\lambda > 0$. The dual space of $\mathcal{D}_\omega(\Omega)$ is denoted by $\mathcal{D}'_\omega(\Omega)$ whose elements are called the generalized distributions on Ω. Also the dual space $\mathcal{E}'_\omega(\Omega)$ of $\mathcal{E}_\omega(\Omega)$ can be identified the set of all elements of $\mathcal{D}'_\omega(\Omega)$ is equal to $\mathcal{D}'$ when $\omega(\xi) = \log(1+|\xi|)$ and $\mathcal{E}_\omega(\Omega)$ is related to the Gevrey class when $\omega(\xi) = |\xi|^{\frac{1}{d}}$, $d > 1$. Almost all results in the distribution theory can be extended to the generalized distribution spaces. For instance, $\mathcal{E}'_\omega * \mathcal{D}'_\omega \subset \mathcal{D}'_\omega$ and the Paley–Wiener–Schwartz theorem in this spaces can be shown as follows :

Let K be a compact convex set in $\mathbf{R}^n$ with support function H. The Fourier–Laplace transform of $\phi \in \mathcal{D}_\omega(K)$ is the entire function $\hat{\phi}(\zeta)$ in C^n such that for each $\lambda > 0$ and each $\varepsilon > 0$, there exists a constant

$C_{\lambda,\varepsilon}$ satisfying $|\hat{\phi}(\xi+i\eta)| \leq C_{\lambda,\varepsilon} e^{H(\eta)+\varepsilon|\eta|-\lambda\omega(\xi)}$ and also the converse is true. On the other hand, the Fourier–Laplace transform of $u \in \mathcal{E}'_\omega$ with supp $u \subset K$ is the entire function $\hat{u}(\zeta)$ in C^n such that for each $\varepsilon > 0$, there exist $\lambda > 0$ and C satisfying $|\hat{u}(\zeta)| \leq C\, e^{H(\eta)+\varepsilon|\eta|+\lambda\omega(\xi)}$ and also true for the converse.

2. Hypoelliptic Convolution Equations

We consider the convolution equation of the form

$$S * u = v \tag{1}$$

where $S \in \mathcal{E}'_\omega$ and $u, v \in \mathcal{D}'_\omega$. We define the convolution equation (1) or the convolutor S is hypoelliptic in $\mathcal{D}'_\omega$ if all solution u of (1) are in $\mathcal{E}_\omega$ whenever v does. Of course, this definition coincides with that in $\mathcal{D}'$ when $\omega(\xi) = \log(1 + |\xi|)$. We obtain the necessary and sufficient condition of hypoellipticity of (1) in $\mathcal{D}'_\omega$. Our main result is

THEOREM. *S is hypoelliptic in $\mathcal{D}'_\omega$ if and only if there is $B > 0$ such that for every positive m there exists C_m such that*

$$|\hat{S}(\zeta)|^{-1} \leq e^{B\,\omega(\mathrm{Re}\,\zeta)+H(-\mathrm{Im}\,\zeta)} \tag{2}$$

if $|Im\,\zeta| \leq m\omega(Re\,\zeta)$ and $|\zeta| \geq C_m$. Here H is the supporting function for supp(S).

We remark that, in view of Hörmander's result [5], our result is equivalent to the Ehrenpreis' one when $\omega(\xi) = \log(1 + |\xi|)$ and to the Chou's one in the ultradistributions of Beurling type when $\omega(\xi) = |\xi|^{\frac{1}{d}}$. $d > 1$.

Before proving the theorem we define that $F \in \mathcal{S}'_\omega$ is a ω–parametrix of S is there exist a compact set K and a function ψ in $\mathcal{E}_\omega$ such that $F \in \mathcal{E}_\omega(K^c)$ and $S * F + \psi = \delta$.

LEMMA. *If S has a ω–parametrix F, then S is hypoelliptic in $\mathcal{D}'_\omega$.*

Proof. Let the compact set K and $\psi \in \mathcal{E}_\omega$ be given in the definition of ω–parametrix F. Take ϕ in $\mathcal{D}_\omega$ with $\phi = 1$ in a neighborhood of K

and let $\tilde{F} = \phi F$. Then $\tilde{F} \in \mathcal{E}_\omega$. Using the fact that $\mathcal{E}'_\omega * \mathcal{E}_\omega \subset \mathcal{E}_\omega$,

$$\begin{aligned} S * \tilde{F} &= S * F + S * (\tilde{F} - F) \\ &= \delta - \psi + S * (\tilde{F} - F) \\ &= \delta + \tilde{\psi} \end{aligned}$$

where $\tilde{\psi} = S * (\tilde{F} - F) - \psi = S * \tilde{F} - \delta \in \mathcal{D}_\omega$. Then

$$\begin{aligned} u = \delta * u &= \{(S * \tilde{F}) - \tilde{\psi}\} * u \\ &= S * \tilde{F} * u - \tilde{\psi} * u \\ &= \tilde{F} * v - \tilde{\psi} * u. \end{aligned}$$

In view of $\mathcal{D}'_\omega * \mathcal{D}_\omega \subset \mathcal{E}_\omega$, the last terms are in $\mathcal{E}_\omega$, which shows our result.

The proof of the theorem: (sufficiency). By the lemma it suffices to show that S has a ω–parametrix F. Let m be a sufficiently large integer which will be chosen later. Denote

$$\hat{F}(\xi) = \begin{cases} \hat{S}(\xi)^{-1} & \text{if } \xi \in \mathbf{R}^n \text{ and } |\xi| \geq C_m \\ 0 & \text{otherwise.} \end{cases}$$

Then $\hat{F} \in \mathcal{S}'_\omega$. In fact, for $\phi \in \mathcal{S}_\omega$,

$$\begin{aligned} |\hat{F}(\phi)| &\leq \int |\hat{F}(\xi)|\,|\phi(\xi)| d\xi \\ &\leq \int_{|\xi| \geq C_m} |\hat{S}(\xi)|^{-1} |\phi(\xi)| d\xi \\ &\leq \int_{|\xi| \geq C_m} e^{B\,\omega(\xi)} |\phi(\xi)| d\xi \\ &\leq \sup_{\xi \in \mathbf{R}^n} \left\{ e^{(B + \frac{n+1}{b})\omega(\xi)} \phi(\xi) \right\} \int_{\mathbf{R}^n} c^{-\frac{n+1}{b}\omega(\xi)} d\xi \\ &\underset{(\gamma)}{\leq} C \sup_{\xi \in \mathbf{R}^n} \left\{ e^{(B + \frac{n+1}{b})\omega(\xi)} \phi(\xi) \right\}. \end{aligned}$$

From the identity $(S * \hat{F}) = \hat{S}\hat{F} = 1 = \chi_{\{|\xi|<C_m\}}$, we have

$$S * F = \delta - (2\pi)^{-n} \int_{|\xi|<C_m} e^{i<x,\xi>} d\xi = \delta - \psi(x).$$

where $\psi(x)$ is an analytic function in $\mathbf{R}^n$.

It remains to show that $F \in \mathcal{E}_\omega(K^c)$ for $K = \{x : |x| \le C_m\} \supset \operatorname{supp} S$ for sufficiently large C_m, which is equivalent to show that for every $x_0 \notin K$, there exists a neighborhood U_{x_0} of x_0 such that $F \in \mathcal{E}_\omega(U_{x_0})$. Since $-x_0 \notin K$, we can find a non-zero $\eta \in \mathbf{R}^n$ so that $< -x_0, \eta >> H_K(\eta)$, where H_K is the supporting function of K. Multiplying η by a constant, we may assume that $H_K(\eta) + < x_0, \eta >< -2$. In what follows, we keep η fixed.

Take a neighbourhood of x_0 by $U_{x_0} = \{x \in \mathbf{R}^n : H(\eta) + < x, \eta >< -2$ and $|x| > C_m\}$. We have to show that $\phi F \in \mathcal{D}_\omega$ for every $\phi \in \mathcal{D}_\omega(U_{x_0})$, that is, $\sup_{\tau \in \mathbf{R}^n} |\widehat{\phi F}(\tau)| e^{\lambda\omega(\tau)} < \infty$ for every $\lambda > 0$.

$$\begin{aligned} \widehat{\phi F}(\tau) &= (\phi F)(e^{-i<x,\tau>}) \\ &= F(\phi(x) e^{-i<x,\tau>}) \\ &= (2\pi)^{-n} \hat{F}((\phi e^{-i<x,\tau>})^{\wedge}) \\ &= (2\pi)^{-n} \int_{|\xi| \ge C_m} \frac{\hat{\phi}(-\xi+\tau)}{\hat{S}(\xi)} d\xi \end{aligned}$$

To evaluate this integral, for every $\xi \in \mathbf{R}^n$ with $|\xi| \ge C_m$. We denote $t(\xi) = m\omega(\xi)/|\eta|$ and $\gamma_m = \{\zeta \in C^n : \zeta = \xi - i\, t(\xi)\eta,\ |\xi| \ge C_m\}$. Using Cauchy theorem we can wirte

$$\begin{aligned} (3) \qquad & \int_{|\xi| \ge C_m} \frac{\hat{\phi}(-\xi+\tau)}{\hat{S}(\xi)} d\xi \\ &= \int_{|\xi| = C_m} \frac{\hat{\phi}(-\xi+\tau+it\eta)}{\hat{S}(\xi - it\eta)} (i\eta) dt + \int_{\gamma_m} \frac{\hat{\phi}(-\zeta+\tau)}{\hat{S}(\zeta)} d\zeta. \end{aligned}$$

In fact, for $\zeta = \xi - it\eta$ and ξ fixed with $|\xi| > C_m$,

$$\begin{aligned}
&|\int_0^{t(\xi)} \frac{\hat\phi(-\zeta+\tau)}{\hat S(\zeta)}(i\eta)dt| \\
&\leq C\int_0^{t(\xi)} e^{\{B\omega(\xi)+H_S(t\eta)\}}e^{\{\lambda\omega(-\xi+\tau)+H_\phi(t\eta)+\varepsilon|t\eta|\}}|\eta|dt \\
&\underset{(\alpha,\delta)}{\leq} C\int_0^{\infty} \exp\{(-\lambda+B)\omega(\xi)+t(H_S(\eta)+\varepsilon|\eta|\}dt
\end{aligned}$$

Since $H_S(\eta)+H_\phi(\eta) \leq H_K(\eta)+\sup_{x\in U_{x_0}} < x,\eta >< -2$, we can take $\varepsilon > 0$ so small that $H_S(\eta)+H_\phi(\eta)+\varepsilon|\eta| < -1$. Hence above integral is finite for each fixed ξ with $|\xi| \geq C_m$ and converges to zero as $|\xi| \to \infty$ by the Lebesque dominate convergence theorem for $\lambda > B$, which implies that the change of the contour in (3) is legitimate.

We now estimate the integral (3).

$$\begin{aligned}
&|(2\pi)^{-n}\int_{|\xi|\geq C_m} \frac{\hat\phi(-\xi+\tau)}{\hat S(\xi)}d\xi| \\
\leq &(2\pi)^{-n}\int_{|\xi|=C_m}^{m\omega(\xi)/|\eta|} \frac{|\hat\phi(-\xi+\tau+it\eta|}{|\hat S(\xi-it\eta)|}|\eta|dt \\
&+(2\pi)^{-n}\int_{\gamma_m} \frac{|\hat\phi(-\zeta+\tau)|}{|\hat S(\zeta)|}|d\zeta| \\
\leq &C\int_{|\xi|=C_m}^{m\omega(\xi)/|\eta|} e^{\{-\lambda\omega(-\xi+\tau)+H_\phi(t\eta)+\varepsilon|t\eta|+B\omega(\xi)+H_S(t\eta)\}}dt \\
&+C\int_{\gamma_m} e^{\{-\lambda\omega(-\xi+\tau)+H_\phi(t(\xi)\eta)+|t(\xi)\eta|+B_{\omega(\xi)}+H_S(t(\xi)\eta)\}}|d\zeta| \\
&\underset{(\alpha,\delta)}{\leq} C e^{-\lambda\omega(\tau)}\left[\int_{|\xi|=C_m}^{m\omega(\xi)/|\eta|} e^{\{(\lambda+B)\omega(\xi)+t(H_S(\eta)+H_\phi(\eta)+\varepsilon|\eta|)\}}dt\right. \\
&\left.+\int_{\gamma_m} e^{\{(\lambda+B)\omega(\xi)+\frac{m\omega(\xi)}{|\eta|}(H_S(\eta)+H_\phi(\eta)+\varepsilon|\eta|)\}}|d\zeta|\right]
\end{aligned}$$

Since $H_S(\eta)+H_\phi(\eta)+\varepsilon|\eta| < -1$ for sufficiently small $\varepsilon > 0$ and $\lambda > 0$

is given, taking m so large that

$$\lambda + B + \frac{m}{|\eta|}(H_S(\eta) + H_\phi(\eta) + \varepsilon|\eta|) < -1,$$

we show that the last integral in bracket is finite. Therefore we have shown that $\sup_{\tau \in \mathbf{R}^n} |\widehat{\phi F}(\tau)| e^{\lambda\omega(\tau)} < \infty$ for every $\phi \in \mathcal{D}_\omega(U_{x_0})$ and $\lambda > 0$, which gives $F \in \mathcal{E}_\omega(K^c)$.

(Necessity) Suppose that the condition (2) is false. Then for every $l > 0$, there exists a positive number m_0, independent of l, and a sequence $\zeta_j = \xi_j + i\eta_j \in C^n$ with $\eta_j \le m_0\omega(\xi_j)$ and $\frac{1}{2}|\xi_{j+1}| \ge |\xi_j| \ge 2^{2j}$, $j = 1, 2, \ldots$ such that

$$|\hat{S}(\zeta_j)|^{-1} \ge e^{l\omega(\xi_j) + H(-\eta_j)}.$$

It suffices to show that there exists a $u \in \mathcal{D}'_\omega$ such that $S * u \in \mathcal{E}_\omega$ and $u \notin \mathcal{E}_\omega$, i.e., S is not hypoelliptic in $\mathcal{D}'_\omega$. We will show that

$$u(x) = \sum_{k=1}^{\infty} e^{i<x,\zeta_k>}$$

satisfies all the requirements.

Using the fact that $\phi_j \to 0$ in $\mathcal{D}_\omega$ is equivalent to

$$\|\phi_j\|_\lambda = \sup_{\zeta \in C^n} |\hat{\phi}_j(\zeta)| e^{\lambda\omega(\xi_j) - H(\eta) - |\eta|} \to 0 \text{ as } j \to \infty,$$

for every $\lambda > 0$, we have

$$\begin{aligned}
|u(\phi_j)| &\le \sum_k |\hat{\phi}_j(-\zeta_k)| \\
&\le \sum_k |\hat{\phi}_j(-\zeta_k)| e^{\lambda\omega(\xi_k) - H_{\phi_j}(\eta_k) - |\eta_k|} e^{-\lambda\omega(\xi_k) + H_{\phi_j}(\eta_k) + |\eta_k|} \\
&\le \|\phi_j\|_\lambda \sum_k e^{-\lambda\omega(\xi_k) + (C+1)m_0\omega(\xi_k)} \\
&\le C\|\phi_j\|_\lambda \to \text{ as } j \to \infty, \text{ for large } \lambda > 0,
\end{aligned}$$

which implies $u \in \mathcal{D}'_\omega$.

To show $S * u \in \mathcal{E}_\omega$, it is enough to show that $\phi(S * u) \in \mathcal{D}_\omega$ or equivalently

$$\sup_{\zeta \in C^n} |\phi(\widehat{S * u})(\zeta)| e^{\lambda\omega(\xi) - H_\phi(\eta) - |\eta|} < \infty$$

for every $\lambda > \mathcal{D}_\omega$. From the equality

$$\begin{aligned} \phi(\widehat{S * u})(\zeta) &= (S * u)_x(\phi(x) e^{-i<x,\zeta>}) \\ &= S_x(\sum_j e^{i<y,\zeta_j>}, \phi(x+y) e^{-i<x+y,\zeta>}) \\ &= \sum_j \hat{\phi}(\zeta - \zeta_j)\hat{S}(\zeta_j), \end{aligned}$$

we have

$$\begin{aligned} &\sup_{\zeta \in C^n} |\phi(\widehat{S * u})(\zeta)| e^{\lambda\omega(\xi) - H(\eta) - |\eta|} \\ &\leq \sum_j \sup_{\zeta \in C^n} |\hat{S}(\zeta_j)|\,|\hat{\phi}(\zeta - \zeta_j)| e^{\lambda\omega(\xi) - H(\eta) - |\eta|} \\ &\leq C_l \sum_j \sup_{\zeta \in C^n} \{e^{-l\omega(\xi_j) - H(-\eta_j)} e^{-\lambda\omega(\xi - \xi_j) + H_\phi(\eta - \eta_j) + \varepsilon(\eta - \eta_j)}\} \\ &\qquad \times e^{\lambda\omega(\xi) - H(\eta) - |\eta|} \\ &\leq C_l \sum_j \exp\{-l\omega(\xi_j) + \lambda\omega(\xi_j) + |\eta_j| - H(-\eta_j) + H_\phi(\eta_j)\} \\ &\leq C_l \sum_j \exp\{(-l + \lambda + (1 + C)m_0)\omega(\xi_j)\} < \infty \end{aligned}$$

for sufficiently large l, which shows $S * u \in \mathcal{E}_\omega$.

It remains to show that $u \notin \mathcal{E}_\omega$. Take $\phi_0 \in \mathcal{D}_\omega$ with $\hat{\phi}_0(0) = 1$ and let $\hat{\psi}_k(\zeta) = k\hat{\phi}_0(\zeta - \zeta_k)$. Then

$$\begin{aligned} |\hat{\psi}_k(\zeta)| &= |k\hat{\phi}_0(\zeta - \zeta_k)| \\ &\leq C_{\lambda,\varepsilon} k e^{\{-\lambda\omega(\xi - \xi_k) + H_{\phi_0}(\eta - \eta_k) + \varepsilon|\eta - \eta_k|\}} \\ &\leq C_{\lambda,\varepsilon} k e^{\{-\lambda\omega(\xi_k) + (C+\varepsilon)m_0\omega(\xi_k) + \lambda\omega(\xi) + H_{\phi_0}(\eta) + \varepsilon|\eta|\}} \\ &\leq C'_{\lambda,\varepsilon} e^{\{\lambda\omega(\xi) + H_{\phi_0}(\eta) + \varepsilon|\eta|\}}, \end{aligned}$$

which gives, from the Paley–Wiener–Schwertz theorem in $\mathcal{D}'_\omega$, that $\psi_k \in \mathcal{E}'_\omega$ for each $k = 1, 2, \ldots,$. Moreover we can show that the set $\{\psi_k\}_1^\infty$ is bounded in $\mathcal{E}'_\omega$. In fact, since $\mathrm{supp}(\psi_k) \subset \mathrm{supp}(\phi)$, we may consider the value of $\psi_k(\phi)$ for each $\phi \in \mathcal{D}_\omega$ instead of $\phi \in \mathcal{E}_\omega$. For $\phi \in \mathcal{D}_\omega$,

$$\begin{aligned}
|\psi_k(\phi)| &= (2\pi)^{-n}|\hat{\psi}_k(\hat{\phi})| \\
&= (2\pi)^{-n}k|\int \hat{\phi}_0(\xi - \zeta_k)\hat{\phi}(\xi)d\xi| \\
&\leq C_{\lambda,\varepsilon} k \int e^{\{-\lambda\omega(\xi-\xi_k)+H(\eta_k)+\varepsilon|\eta_k|-2\lambda\omega(\xi)\}} d\xi \\
&\leq C_{\lambda,\varepsilon} k e^{\{-\lambda\omega(\xi_k)+(C_0+\varepsilon)m_0\omega(\xi_k)\}} \int e^{-\lambda\omega(\xi)} d\xi \\
&\leq \tilde{C}_{\lambda,\varepsilon} k e^{\{-\lambda'\omega(\xi_k)\}} \\
&\underset{(\gamma)}{\leq} \tilde{C}_{\lambda,\varepsilon} \frac{k}{(1+|\xi_k|)^{\lambda' b}} \\
&\leq \tilde{C}_{\lambda,\varepsilon} \frac{k}{(1+2^{2k})^{\lambda' b}} \\
&\leq \tilde{C}_{\lambda,\varepsilon} < \infty
\end{aligned}$$

for all $k = 1, 2, \ldots$, which shows our claim. Finally we have to show that $\{u(\psi_k)\}_1^\infty$ is not bounded. We can write

$$\begin{aligned}
|u(\psi_k)| &= |\sum_j (e^{i<x,\zeta_j>},\ \psi_k(x)| \\
&= |\sum_j \hat{\psi}_k(\zeta_j)| \geq |\hat{\psi}_k(\zeta_k)| - \sum_{j\neq k} |\hat{\psi}_k(\zeta_j)| \\
&= k - \sum_{j\neq k} |\hat{\psi}_k(\zeta_j)| \to \infty \text{ as } k \to \infty,
\end{aligned}$$

provided that $\sum_{j\neq k} |\hat{\psi}_k(\zeta_j)|$ is uniformly bounded on k.
Therefore it suffices to show that $\sum_{j\neq k} |\hat{\psi}_k(\zeta_j)|$ is bounded uniformly for $k = 1, 2, \ldots$. Dividing the sum into two pieces for $j < k$ and $j > k$,

$$|\xi_j| < \frac{1}{2^{(k-j)}}|\xi_k| \text{ and } |\xi_j - \xi_k| \geq |\xi_k| - |\xi_j| > \frac{2^{(k-j)} - 1}{2^{(k-j)}}|\xi_k|$$

when $j < k$.
With the notation $\xi_j' = (1 - \frac{1}{2^{(k-j)}})\xi_k$ and $\xi_j'' = \frac{1}{2^{(k-j)}}\xi_k$, we have, from $|\eta_j| \leq m_0\omega(\xi_j)$ and $\xi_j' \geq \xi_j''$,

$$\begin{aligned}
&\sum_{j<k} |\hat{\psi}_k(\zeta_j)| \\
\leq & C_{\lambda,\varepsilon} \sum_{j<k} k \exp\{-\lambda\omega(\xi_j - \xi_k) + (C+\varepsilon)m_0(\omega(\xi_k) + \omega(\xi_j))\} \\
\leq & C_{\lambda,\varepsilon} \sum_{j<k} k \exp\{-\lambda\omega(\xi_j') + (C+\varepsilon)m_0(\omega(\xi_k) + \omega(\xi_j''))\} \\
\leq & C_{\lambda,\varepsilon} \sum_{j<k} k \exp\{(-\lambda + (C+\varepsilon)m_0)\omega(\xi_j') + 2(C_0+\varepsilon)m_0\omega(\xi_j'')\} \\
\leq & C_{\lambda,\varepsilon} \sum_{j<k} k \exp\{(-\lambda + 3(C+\varepsilon)m_0)\omega(\xi_j'')\} \\
\leq & C'_{\lambda,\varepsilon} \sum_{j<k} \frac{k}{(1+|\xi_j''|)^{\{-\lambda+3(C+\varepsilon)m_0\}b}} \\
\leq & C'_{\lambda,\varepsilon} \sum_{j<k} \frac{k(2^{(k-j)})^{\{-\lambda+3(C+\varepsilon)m_0\}b}}{(2^{(k-j)} + 2^{2k})^{\{-\lambda+3(C+\varepsilon)m_0\}b}}
\end{aligned}$$

which is uniformly bounded on k for sufficiently large λ. In the case when $j > k$, using the notation

$$\xi_j^i = (1 - \frac{1}{2^{(j-k)}}\xi_j, \; \xi_j'' = \frac{1}{2^{(j-k)}}\xi_j,$$

$$|\xi_j - \xi_k| \geq |\xi_j| - |\xi_k| \geq (1 - \frac{1}{2^{(j-k)}}\xi_j \geq \frac{1}{2^{(j-k)}}|\xi_j|$$

and

$$|\xi_k| \leq \frac{1}{2^{(j-k)}}|\xi_j|,$$

we have the following estimation in the same way :

$$
\begin{aligned}
&\sum_{j>k} |\hat{\psi}_k(\zeta_j)| \\
\leq & C_{\lambda,\varepsilon} \sum_{j>k} k \exp\{-\lambda\omega(\xi_j - \xi_k) + (C+\varepsilon)m_0(\omega(\xi_j) + \omega(\xi_k))\} \\
\leq & C_{\lambda,\varepsilon} \sum_{j>k} k \exp\{(-\lambda + 3(C+\varepsilon)m_0)\omega(\xi_j'')\} \\
\leq & C'_{\lambda,\varepsilon} \sum_{j>k} \frac{k}{(1+|\xi_j''|)^{\{-\lambda+3(C+\varepsilon)m_0\}b}} \\
\leq & C'_{\lambda,\varepsilon} \sum_{j>k} \frac{k(2^{(j-k)})^{\{-\lambda+3)(C+\varepsilon)m_0\}b}}{(2^{(j-k)} + 2^{2j})^{\{-\lambda+3(C+\varepsilon)m_0\}b}}
\end{aligned}
$$

which is uniformly bounded on k for sufficiently large $\lambda > 0$. Hence $\sum_{j\neq k} |\hat{\psi}_k(\xi_j)|$ is uniformly bounded on k.

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