

SEMI-LINEARITY OF CYCLIC ACTIONS ON SOME LOW DIMENSIONAL SPHERES

D.Y.KWAK AND DONG YOUP SUH

§1. Introduction

Let V be an orthogonal representation of a group G . Then the unit sphere $S(V)$ has the induced action of G , and such action is called a linear action. It is a fundamental question in transformation group theory to ask how similar arbitrary actions on spheres are to linear actions. One of classical results to answer this question is the following theorem of P.A. Smith.

THEOREM 1 [3.5.1, Br] *If G is a p -group (p , prime) and if K is a finite dimensional regular G -complex which is a mod p homology n -sphere [i.e. $H(K; \mathbf{Z}_p) \approx H(S^n; \mathbf{Z}_p)$], then the fixed point set K^G is a mod p homology r -sphere for some $-1 \leq r \leq n$ (where $r = -1$ means that K^G is empty). If p is odd, then $n - r$ is even.*

Since theorem 1 holds trivially for arbitrary linear actions of any group theorem 1 shows that any p -group actions on mod p homology n -spheres are equivalent to some linear actions in mod p homological sense.

In this paper we consider smooth cyclic G actions on integral homology spheres \sum^d ($d \leq 6$) with two fixed points $\sum^G = \{x, y\}$, and study how close they are to linear actions on unit spheres $S(V)$ of a representation V of G .

Suppose a group G acts (smoothly) on an integral homology sphere. We say that the action is semi-linear if the fixed point set \sum^H is an integral homology sphere for each subgroup H of G . If each fixed point set \sum^H is a rational homology sphere, we say that the action is rationally semi-linear. The following theorem is our fundamental result.

THEOREM A. *Suppose a cyclic group G acts smoothly and effectively on an integral homology sphere Σ^d with two fixed points $\Sigma^G = \{x, y\}$.*

- (i) *If $d \leq 4$, or $d = 5$ and n is divisible by 4, then the action of G on Σ is semi-linear.*
- (ii) *When $d = 5$ and $n = 2k$ with $k = \text{odd}$, or $d = 6$, assume Σ^H is connected whenever $\Sigma^H \neq \{p, q\}$. Then the action of G on Σ is rationally semi-linear.*

One might ask if any cyclic actions on homology spheres are semi-linear. But the answer is no and one of such results is as follows: Dovermann and Petrie [DP] (resp. Dovermann and Suh [DS]) constructed examples of homotopy spheres Σ with actions of cyclic groups $G = \mathbf{Z}_n$ (resp. \mathbf{Z}_{2n}), where n is odd, such that $\Sigma^G = \{p, q\}$ and their isotropy representations $T_p \Sigma$ and $T_q \Sigma$ of G are not isomorphic. On the other hand Sanchez [Sa] showed that if the action of G is rationally semi-linear, then those isotropy representations $T_p \Sigma$ and $T_q \Sigma$ are isomorphic. Hence G spheres constructed in [DP] and [DS] are not rationally semi-linear. However the following question is still worthwhile to think about :

QUESTION: What is the highest dimension of homology sphere Σ so that any cyclic action on Σ with two fixed points are rationally semi-linear?

§2. Proof of theorem A

Suppose a group G acts smoothly on a manifold M . If $x \in M^G$, then the action of G around x induces a linear action on the tangent space $T_x M$ at x , which makes $T_x M$ a real representation of G . Such representations are called isotropy representations of G at x .

Let $\mathbf{Z}_n = \{g^k \mid g = \exp \frac{2\pi i}{n}, 0 \leq k \leq n-1\}$ denote the cyclic group of order n . Let t^s denote the complex 1-dimensional representation of \mathbf{Z}_n such that $g \cdot z = \exp \frac{2\pi si}{n} \cdot z$ for $z \in t^s$, where the right-hand side of the equation is the complex multiplication. As a real representation t^s is irreducible and isomorphic to t^{n-s} for $1 \leq s \leq [\frac{n}{2}]$. Here $[\frac{n}{2}]$ is the greatest integer less than or equal to $\frac{n}{2}$. If n is even, $t^{\frac{n}{2}}$ is not irreducible

and isomorphic to $2\mathbf{R}_-$. Here \mathbf{R}_- is the nontrivial real 1-dimensional representation of \mathbf{Z}_n where $g = \exp \frac{2\pi i}{n}$ acts on $x \in \mathbf{R}_-$ by $g \cdot x = x$.

We need the following well-known version of the Lefschetz fixed point theorem.

LEMMA 2. *Let G be a finite cyclic group, and let X be a finite G CW complex. For a generator $g \in G$ its Lefschetz number*

$$\lambda(g) = \chi(X^g)$$

where $\chi(X^g)$ is the Euler characteristic of $X^g = X^G$.

Let X be a G space. For $x \in X$ its isotropy subgroup G_x is defined by $G_x = \{g \in G \mid gx = x\}$. Let $\text{Iso}(X)$ denote the collection of all isotropy subgroups

$$\text{Iso}(X) = \{G_x \mid x \in X\}.$$

We need the following two lemmas

LEMMA 3. *Let a group G act smoothly on a manifold Σ with two fixed points $\Sigma^G = \{x, y\}$. If Σ^H is either connected or two points for each subgroup $H \subset G$, then*

$$\text{Iso}(T_x \Sigma) = \text{Iso}(\Sigma) = \text{Iso}(T_y \Sigma).$$

Proof. Let $H \in \text{Iso}(T_x \Sigma)$. Then $H = G_a$ for some $a \in \Sigma$ near x . Thus $a \in \Sigma^H$. If $a \neq x$, then from the hypothesis Σ^H is connected. Since $y \in \Sigma^H$ there is a point $b \in \Sigma$ around y such that $b \in \Sigma^H$. Thus $G_a \subset G_b$. Apply the same argument to G_b to get $G_a \supset G_b$. Hence $\text{Iso}(T_x \Sigma) = \text{Iso}(T_y \Sigma)$. Since $\text{Iso}(T_x \Sigma) \subset \text{Iso}(\Sigma)$ it remains to prove $\text{Iso}(\Sigma) \subset \text{Iso}(T_x \Sigma)$. Given $G_a \in \text{Iso}(\Sigma)$ if a is neither x nor y , then Σ^{G_a} is connected. Thus Σ^{G_a} is a $N(G_a)/G_a$ manifold where $N(G_a)$ is the normalizer of G_a . Since $x \in \left(\Sigma^{G_a}\right)^{N(G_a)/G_a} = \Sigma^{N(G_a)}$ there exists a point $z \in \Sigma^{G_a}$ near x whose isotropy subgroup of $N(G_a)/G_a$ is trivial. Then $G_z \in \text{Iso}(T_x \Sigma)$ and $G_z = G_a$. Thus $\text{Iso}(\Sigma) \subset \text{Iso}(T_x \Sigma)$.

The following lemma is well known.

LEMMA 4. *If G acts smoothly on a manifold Σ , then for any subgroup $H \subseteq G$ there exists an isotropy subgroup $G_a \in \text{Iso}(\Sigma)$ such that $\Sigma^H = \Sigma^{G_a}$.*

Lemma 3 and lemma 4 show that in order to prove theorem A it is enough to show that Σ^H is a rational homology sphere for each $H \in \text{Iso}(T_x \Sigma)$.

PROOF OF THEOREM A: We treat two cases, $d = \text{odd}$ and $d = \text{even}$, separately.

(A) $d = \text{odd}$.

In this case isotropy representations of G at x and y must contain \mathbf{R}_- as a subrepresentation, and hence n is even. Moreover the index 2 subgroup $\mathbf{Z}_{\frac{n}{2}}$ must be an isotropy subgroup of Σ .

(I) $d = 1$

In this case only $G = \mathbf{Z}_2$ can act effectively on $\Sigma^1 = S^1$ with two fixed points because isotropy representations $T_x \Sigma$ and $T_y \Sigma$ at x and y are

$$T_x \Sigma = T_y \Sigma = \mathbf{R}_-$$

Thus the action of G is trivially semi-linear.

(II) $d = 3$

Let $T_x \Sigma = t^a + \mathbf{R}_-$ be the isotropy representation of G at x . Let $\alpha = \text{g. c. d}(n, a)$. Then $\text{Iso}(t^a) = \{\mathbf{Z}_\alpha, G\}$ where $\text{Iso}(t^a)$ is the collection of all isotropy subgroups of t^a . Hence

$$\text{Iso}(\Sigma) = \text{Iso}(T_x \Sigma) = \{1, \mathbf{Z}_\alpha \cap \mathbf{Z}_{\frac{n}{2}}, \mathbf{Z}_\alpha, \mathbf{Z}_{\frac{n}{2}}, G\}$$

In order that the action is effective either \mathbf{Z}_α is trivial or \mathbf{Z}_α is the 2-Sylow subgroup of G . If \mathbf{Z}_α is the 2-Sylow subgroup of G , then $\mathbf{Z}_\alpha \cong \mathbf{Z}_2$ otherwise the action is not effective, and $\mathbf{Z}_\alpha \cap \mathbf{Z}_{\frac{n}{2}} = 1$. Hence

$$\begin{aligned} \Sigma^{\mathbf{Z}_\alpha} &= \begin{cases} S^2 & \text{if } \mathbf{Z}_\alpha \cong \mathbf{Z}_2 \\ \Sigma & \text{if } \mathbf{Z}_\alpha = 1 \end{cases} \\ \Sigma^{\mathbf{Z}_{\frac{n}{2}}} &= S^1, \end{aligned}$$

and this shows that the action is semi-linear.

(III) $d = 5$

Let $T_x \sum = t^a + t^b + \mathbf{R}_-$, and let g. c. $d(n, a) = \alpha$ and g. c. $d(n, b) = \beta$. In order that the action is effective

$$\text{g. c. } d(\alpha, \beta) = \begin{cases} 1 & \text{if } n \text{ is divisible by } 4 \\ 1 \text{ or } 2 & \text{otherwise.} \end{cases}$$

Moreover $\text{g. c. } d(\alpha, \beta) = 2$ only if $\frac{n}{2}$ is odd.

(Case 1) $\text{g. c. } d(\alpha, \beta) = 2$.

Then $\text{Iso}(\sum) = \{1, \mathbf{Z}_2, \mathbf{Z}_\alpha \cap \mathbf{Z}_{\frac{n}{2}}, \mathbf{Z}_\beta \cap \mathbf{Z}_{\frac{n}{2}}, \mathbf{Z}_{\frac{n}{2}}, G\}$. By theorem 1 the fixed point set $\sum^{\mathbf{Z}_2}$ is a mod 2 homology 4-sphere. For any prime power order subgroup P of $\mathbf{Z}_\alpha \cap H$ its fixed point set \sum^P is a mod p homology 3-sphere. Again by the dimension argument $\sum^{\mathbf{Z}_\alpha \cap \mathbf{Z}_{\frac{n}{2}}} = \sum^P$ is a rational homology 3-sphere. Similarly $\sum^{\mathbf{Z}_\beta \cap \mathbf{Z}_{\frac{n}{2}}}$ is a rational homology 3-sphere. If $\alpha = \beta = 2$, then $\sum^{\mathbf{Z}_\alpha} = \sum^{\mathbf{Z}_2}$ is a mod 2 homology 4-sphere. If $\alpha \neq \beta$, then the dimension of $\sum^{\mathbf{Z}_\alpha} = 2$ from the hypothesis of the theorem $\sum^{\mathbf{Z}_\alpha}$ is connected. Let $f \in \mathbf{Z}_\alpha$ be a generator of \mathbf{Z}_α . Then $f : \sum \rightarrow \sum$ is an orientation reversing map. Thus $\lambda(f) = 2$. By lemma 2 the Euler characteristic $\chi(\sum^{\mathbf{Z}_\alpha}) = 2$. Since $\sum^{\mathbf{Z}_\alpha}$ is a surface $\sum^{\mathbf{Z}_\alpha} = S^2$. Similarly $\sum^{\mathbf{Z}_\beta} = S^2$. $\sum^{\mathbf{Z}_{\frac{n}{2}}} = S^1$, hence the action is rationally semi-linear.

(Case 2) $\text{g. c. } d(\alpha, \beta) = 1$.

Note that if both α and β are not equal to 1, then exactly one of them, say α , must be divisible by 2, and the 2-Sylow subgroup G_2 of G must be a subgroup of \mathbf{Z}_α . If $\alpha = \beta = 1$, then the action is trivially semi-linear. Otherwise, $\sum^{\mathbf{Z}_\alpha} = \sum^{G_2} = S^2$. Since $\mathbf{Z}_\beta \subset \mathbf{Z}_{\frac{n}{2}}$ the fixed point set $\sum^{\mathbf{Z}_\beta \cap \mathbf{Z}_{\frac{n}{2}}} = \sum^{\mathbf{Z}_\beta} = \text{mod } p \text{ homology } 3\text{-sphere}$ for any prime p dividing β . If $\mathbf{Z}_\alpha \cap H \neq 1$, then since $\text{g. c. } d(\alpha, \beta) = 1$ there exists a prime p such that $p|\alpha$ but $p \nmid \beta$. Thus $\sum^{\mathbf{Z}_\alpha \cap H}$ is a mod p homology 3-sphere. If $\mathbf{Z}_\alpha \cap H = 1$, then $\mathbf{Z}_\alpha = \mathbf{Z}_2$ and $\sum^{\mathbf{Z}_\alpha} = S^2$. Again $\sum^{\mathbf{Z}_{\frac{n}{2}}} = S^1$. Thus the action is rationally semi-linear.

(B) d is even.

(I) $d = 2$.

Let $T_x \sum = t^a$. Then $\text{g. c. } d(n, a) = 1$, hence $\text{Iso}(\sum) = \{1, G\}$. Thus the action is trivially semi-linear.

(II) $d = 4$.

Let $T_x \Sigma = t^a + t^b$, and let $\text{g. c. d}(n, a) = \alpha$ and $\text{g. c. d}(n, b) = \beta$. In order that the action is effective $\text{g. c. d}(\alpha, \beta) = 1$. Hence $\text{Iso}(\Sigma) = \{1, \mathbf{Z}_\alpha, \mathbf{Z}_\beta, G\}$. Thus by the same argument as in the previous cases the action is semi-linear.

(III) $d = 6$.

Let $T_x \Sigma = t^a + t^b + t^c$, and let $\text{g. c. d}(n, a) = \alpha$, $\text{g. c. d}(n, b) = \beta$, and $\text{g. c. d}(n, c) = \gamma$. Note that $\text{g. c. d}(\alpha, \beta, \gamma) = 1$.

(Case 1) $\text{g. c. d}(\alpha, \beta) = \text{g. c. d}(\beta, \gamma) = \text{g. c. d}(\alpha, \gamma) = 1$. Then $\text{Iso}(\Sigma) = \{1, \mathbf{Z}_\alpha, \mathbf{Z}_\beta, \mathbf{Z}_\gamma, G\}$.

Since $\sum^{\mathbf{Z}_\alpha} \cong \sum^{\mathbf{Z}_\beta} \cong \sum^{\mathbf{Z}_\gamma} \cong S^2$ the action is semi-linear.

(Case 2) $\text{g. c. d}(\alpha, \beta) = \text{g. c. d}(\alpha, \gamma) = 1$ and $\text{g. c. d}(\beta, \gamma) = \delta \neq 1$. Then $\text{Iso}(\Sigma) = \{1, \mathbf{Z}_\delta, \mathbf{Z}_\alpha, \mathbf{Z}_\beta, \mathbf{Z}_\gamma, G\}$. If $\beta = \gamma$, then $\sum^{\mathbf{Z}_\beta} = \sum^{\mathbf{Z}_\gamma}$ is a rational homology 4-sphere and $\sum^{\mathbf{Z}_\alpha} \cong S^2$. If $\beta \neq \gamma$, then $\sum^{\mathbf{Z}_\alpha}$, $\sum^{\mathbf{Z}_\beta}$, and $\sum^{\mathbf{Z}_\gamma}$ are all homeomorphic to S^2 , and $\sum^{\mathbf{Z}_\delta}$ is a rational homology 4-sphere. Thus the action is rationally semi-linear.

(Case 3) $\text{g. c. d}(\alpha, \beta) = \delta \neq 1$, $\text{g. c. d}(\beta, \gamma) = \varepsilon \neq 1$, and $\text{g. c. d}(\alpha, \gamma) = \eta \neq 1$. Then $\text{Iso}(\Sigma) = \{1, \mathbf{Z}_\alpha, \mathbf{Z}_\beta, \mathbf{Z}_\gamma, \mathbf{Z}_\delta, \mathbf{Z}_\varepsilon, \mathbf{Z}_\eta, G\}$. There are distinct primes p , q , and r such that $p|\delta$, $q|\varepsilon$, and $r|\eta$. Then $\sum^{\mathbf{Z}_\delta} = \sum^{\mathbf{Z}_p} = \text{mod } p \text{ homology 4-sphere}$, $\sum^{\mathbf{Z}_\varepsilon} = \sum^{\mathbf{Z}_q} = \text{mod } q \text{ homology 4-sphere}$, and $\sum^{\mathbf{Z}_\eta} = \sum^{\mathbf{Z}_r} = \text{mod } r \text{ homology 4-sphere}$. Let $g \in \mathbf{Z}_\alpha$ be a generator of \mathbf{Z}_α . Since g is an orientation preserving map on Σ by lemma 2 $\lambda(g) = 2 = \chi(\sum^{\mathbf{Z}_\alpha})$. Since $\dim(\sum^{\mathbf{Z}_\alpha}) = 2$ it is connected. Thus $\sum^{\mathbf{Z}_\alpha} \cong S^2$. Similarly both $\sum^{\mathbf{Z}_\beta}$ and $\sum^{\mathbf{Z}_\gamma}$ are homeomorphic to S^2 . Hence the action is semi-linear.

References

- [Br] G. Bredon, "Introduction to compact transformation groups," Academic press, 1972.
- [DP] K.H. Dovermann and T. Petrie, *Smith equivalence of representations for odd order cyclic groups*, Topology **24** No3 (1985), 283-305.
- [DS] K.H. Dovermann and D.Y. Suh, *Smith equivalence for even order cyclic groups*, preprint (1989).

[Sa] C. Sanchez, *Actions of odd order on compact orientable manifolds.*, Proc. AMS 54 (1976), 445-448.

Department of Mathematics
Korea Institute of Technology
Taejon, 302-338 Korea