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STABILITY IN TOPOLOGICAL DYNAMICS *

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A *flow* on a space X is the triplet (X, G, f), where G is a subgroup of $\mathbf{R}$ and f is a continuous map from the product space $X \times G$ into the space X satisfying the following axioms :

$$f(x,0) = x, \quad \text{and}$$
$$f(f(x,t),s) = f(x,t+s)$$

for every x in X and t, s in G. In particular, the flow (X, G, f) is said to be *discrete* if G is a discrete subgroup of $\mathbf{R}$.

Throughout the paper, X denotes a metric space with a metric d. For any two elements x in X and t in $\mathbf{R}$, $f(x,t)$ will be denoted by xt.

Let $(X, \mathbf{R}, f)$ be a flow. A point x in X is said to be (*positively*) *Lipschitz stable* if there exist $\delta(x) > 0$ and $K(x) \geq 1$ such that

$$d(xt, yt) \leq Kd(x,y)$$

for all $t \in (\mathbf{R}^+)\mathbf{R}$ and all $y \in X$ with $d(x,y) < \delta$. When one can select those $\delta > 0$ and $K \geq 1$ independently of all points x in X, the flow $(X, \mathbf{R}, f)$ is called (*positively*) *Lipschitz stable*. A point x in X is said to be (*positively*) *Liapunov stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x,y) < \delta$ implies $d(xt, yt) < \varepsilon$ for all $t \in (\mathbf{R}^+)\mathbf{R}$. A flow $(X, \mathbf{R}, f)$ is called (*positively*) *Liapunov stable* if every point in X is (positively) Liapunov stable. The negative versions of Lipschitz and Liapunov stability can be defined in a similar way.

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Now we describe some relations between a flow and discrete flows induced from this flow. Let $(X, \mathbf{R}, f)$ be a flow. Then it is clear that for each $\tau \in \mathbf{R}$, the map $f_\tau : X \times \mathbf{Z} \to X$ given by

$$f_\tau(x, n) = f(x, n\tau)$$

is continuous, $f_\tau(x, 0) = x$, and $f_\tau(f_\tau(x, m), n) = f_\tau(x, m + n)$ for each x in X and m, n in $\mathbf{Z}$. Consequently $(X, \mathbf{Z}, f_\tau)$ is a discrete flow, and it will be called the discrete flow induced by τ from $(X, \mathbf{R}, f)$. For brevity, we call $(X, \mathbf{Z}, f_\tau)$ as an induced discrete flow. Then we have a question whether the dynamic properties of a flow $(X, \mathbf{R}, f)$ can be inherited to the induced discrete flow $(X, \mathbf{Z}, f_\tau)$, $\tau \in \mathbf{R}$, and vice versa. Here we show that the Liapunov stability of an induced discrete flow $(X, \mathbf{Z}, f_\tau)$, $\tau \in \mathbf{R}$, can be inherited to the flow $(X, \mathbf{R}, f)$, but the Lipschitz stability cannot be inherited.

First we give an example to show that a flow $(X, \mathbf{R}, f)$ need not be Lipschitz stable even if an induced discrete flow $(X, \mathbf{Z}, f_\tau)$, $\tau \in \mathbf{R}$, is Lipschitz stable, in particular, isometric and the phase space X is compact metric.

EXAMPLE 1. *Let us consider the flow f on the compact space $X = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 \leq 1\}$, generated from the differential system (polar coordinates):*

$$\begin{cases} r^3 \cos^2\theta(r \sin\theta - \frac{r'}{2\pi}\cos\theta) = \sin\theta(r\cos\theta + \frac{r'}{2\pi}\sin\theta)^2, \\ \theta' = 2\pi. \end{cases}$$

Then the orbit $O(a, 0)$ passing through a point $(a, 0)$, $0 < a \leq 1$, in X is the ellipse

$$\{(x, y) \in \mathbf{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{a^4} = 1\},$$

and $(0, 0)$ is the unique critical point of the flow. Furthermore we can see that the point $(0, 0)$ is not Lipschitz stable, and so the flow $(X, \mathbf{R}, f)$ is not Lipschitz stable. On the other hand, let us consider the discrete flow $(X, \mathbf{Z}, f_{\frac{1}{2}})$ induced by $1/2 \in \mathbf{R}$ from $(X, \mathbf{R}, f)$. Then the induced discrete flow $(X, \mathbf{Z}, f_{\frac{1}{2}})$ is isometric.

THEOREM 2. *(Integral Continuity Condition) For any point x in X, any number $T > 0$ and any $\varepsilon > 0$, there exists $\delta > 0$ such that $d(xt, yt) <$*

ε for all $y \in X$ and $t \in \mathbf{R}$ which satisfy the inequalities $d(x, y) < \delta$ and $0 \le t \le T$ $(-T \le t \le 0)$ [6].

Using the above theorem, we obtain the following theorem.

THEOREM 3. *A flow $(X, \mathbf{R}, f)$ is Liapunov stable if and only if there exists an induced discrete flow $(X, \mathbf{Z}, f_\tau)$ which is Liapunov stable for some $\tau > 0$.*

Proof. Assume that an induced discrete flow $(X, \mathbf{Z}, f_\tau)$ is Liapunov stable for some $\tau > 0$, and choose a point p in X. Then for any point y in $p[0, \tau]$ and any $\varepsilon > 0$, there exists $\delta(y) > 0$ such that $d(y, z) < \delta(y)$ implies $d(yn\tau, zn\tau) < \frac{\varepsilon}{2}$ for all $n \in \mathbf{Z}$. Let z_1 and z_2 be any two points in $B(y, \delta(y))$. Since $d(yn\tau, z_1n\tau) < \frac{\varepsilon}{2}$ and $d(yn\tau, z_2n\tau) < \frac{\varepsilon}{2}$, we have $d(z_1n\tau, z_2n\tau) < \varepsilon$. Since $p[0, \tau]$ is compact, there are points $y_1, \ldots, y_m$ in $p[0, \tau]$ such that

$$p\,[0, \tau] \subset \bigcup_{i=1}^{m} B(y_i, \delta(y_i)).$$

Hence there exists $\delta > 0$ such that for every point y in $p[0, \tau]$, $B(y, \delta) \subset B(y_i, \delta(y_i))$ for some $i \in \{1, \ldots, m\}$. By Theorem 2, there exists $\delta' > 0$ such that $d(p, x) < \delta'$ implies $d(pt, xt) < \delta$ for all t in $[0, \tau]$. So we get

$$xt \in B(pt, \delta) \subset B(y_i, \delta(y_i))$$

for some $i \in \{1, \ldots, m\}$. Hence we obtain

$$d(ptn\tau,\ xtn\tau) < \varepsilon$$

for all $n \in \mathbf{Z}$. Let s be any point in $\mathbf{R}$. Then there exists $m \in \mathbf{Z}$ such that

$$(m-1)\tau \le \frac{s}{\tau} \le m\tau.$$

If $m > 0$ then, by letting $t = \frac{s}{m\tau}$, we have $0 \le t \le \tau$ and $s = tm\tau$. So we obtain $d(ps, xs) < \varepsilon$. If $m < 0$, then we similarly have $d(ps, xs) < \varepsilon$. Therefore the point p is Liapunov stable for the flow $(X, \mathbf{R}, f)$. Since the point p is arbitrary in X, the flow $(X, \mathbf{R}, f)$ is also Liapunov stable.

Conversely, it is clear that if the flow $(X, \mathbf{R}, f)$ is Liapunov stable then an induced discrete flow $(X, \mathbf{Z}, f_\tau)$ is Liapunov stable for $\tau > 0$.

Now we will investigate some relations among the dynamic properties (i.e. periodic, almost periodic, almost recurrent, recurrent, Poisson stable, nonwandering). To do this, we introduce those concepts in [2] and [6].

Let $(X, \mathbf{R}, f)$ be a flow. A point p in X is called *almost periodic* if for every $\varepsilon > 0$ there exists a relatively dense subset of numbers $\{\tau_n\}$ such that

$$d(pt,\ p(t+\tau_n)) < \varepsilon$$

for all $t \in \mathbf{R}$ and each τ_n. A point p in X is said to be *almost recurrent* (or *recurrent*) if for any $\varepsilon > 0$ there is a number $T > 0$ such that

$$p \in B(p\,[t,\ t+T], \varepsilon)\ (\text{or}\ \ p\mathbf{R} \subset B(p\,[t,\ t+T], \varepsilon)\,])$$

for any t in $\mathbf{R}$. A point p in X is called *positively* (or *negatively*) *Poisson stable* provided $p \in L^+(p)$ (or $p \in L^-(p)$), and p is said to be (*bilaterally*) *Poisson stable* if it is both positively and negatively Poisson stable. A point p in X is said to be *nonwandering* if $p \in J^+(p)$. If one of the properties above holds at each point of the phase space X, then the flow $(X, \mathbf{R}, f)$ is said to have that property. A set $M \subset X$ is called *minimal* if it is a closed invariant set containing no nonempty proper subset with these properties.

First of all, we show that the concept of almost periodicity and that of almost recurrence are equivalent, under the Liapunov stable flow.

THEOREM 4. *Let $(X, \mathbf{R}, f)$ be a Liapunov stable flow. Then a point in X is almost periodic if and only if it is recurrent [2].*

THEOREM 5. *Let $(X, \mathbf{R}, f)$ be a Liapunov stable flow. Then a point in X is recurrent if and only if it is almost recurrent.*

Proof. Let a flow $(X, \mathbf{R}, f)$ be Liapunov stable, and suppose that a point p in X is almost recurrent. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that $d(p, x) < \delta$ implies $d(pt, xt) < \frac{\varepsilon}{2}$ for all t in $\mathbf{R}$. Given $\delta > 0$, we can choose a number $T > 0$ such that

$$p \in B(p[s, s+T], \delta)$$

for any s in $\mathbf{R}$. Thus there exists a point u in $[0,T]$ satisfying $d(p,p(s+u)) < \delta$. Let t be an arbitrary fixed point in $\mathbf{R}$. Then there is a point v in $[t+u-T,\ t+u]$ such that $d(pv,p) < \delta$. Hence we have

$$d(pv(t-v),p(t-v)) = d(pt,p(t-v)) < \frac{\varepsilon}{2}.$$

Since $p(t-v) \in B(p[s,s+T],\ \frac{\varepsilon}{2})$, we obtain

$$pt \in B(p[s,s+T],\ \varepsilon).$$

Consequently we have

$$p\mathbf{R} \subset B(p[s,s+T],\ \varepsilon).$$

Therefore the point p is recurrent.

EXAMPLE 6. *Let $(T,\mathbf{R},\pi)$ be a flow defined on a torus by means of the planar differential system*

$$dx/dt = g(x,y),\ dy/dt = \alpha g(x,y)$$

where $g(x,y) = g(x+1,y) = g(x,y+1) = g(x+1,y+1), g(x,y) > 0$ if x and y are not both zero $(\mathrm{mod}\,1)$ and $g(0,0) = 0$. Let $\alpha > 0$ be irrational. Then the orbits of this flow consist of a critical point p corresponding to the point $(0,0)$. Also there is exactly one orbit O_1 such that $L^-(O_1) = \{p\}$, and exactly one orbit O_2 such that $L^+(O_2) = \{p\}$. For any other orbit O, $L^+(O) = L^-(O) = T$. Furthermore $L^+(O_1) = L^-(O_2) = T$ and the flow does not satisfy the properties of Liapunov stability in neighborhoods of the critical point p. Moreover we have that the points on O_2 are negatively Poisson stable and nonwandering, but they are not positively Poisson stable. Also the orbit closure $\overline{O_2}$ is not minimal. Consequently we get that every point, except p, in T is not recurrent even if each point on the orbits O and O_2 is negatively Poisson stable.

As we know in Example 6, the positively Poisson stability does not imply the negatively Poisson stability and vice versa even if the phase space is compact metric. Also the concepts of nonwandering property and positively (or negatively) Poisson stability are not equivalent. Then we have a question when these concepts are pairwise equivalent. The following two theorems give an answer for this question.

THEOREM 7. *Let $(X, \mathbf{R}, f)$ be a Liapunov stable flow. Then a point in X is positively Poisson stable if and only if it is negatively Poisson stable.*

Proof. Let x be a positively Poisson stable point in X. Then there exists a sequence $\{t_n\}$ in $\mathbf{R}^+$ such that

$$t_n \longrightarrow \infty \quad \text{and} \quad xt_n \longrightarrow x.$$

Since the flow $(X, \mathbf{R}, f)$ is Liapunov stable, we obtain $x(-t_n) \to x$. This implies that the point x is negatively Poisson stable.

THEOREM 8. *Let $(X, \mathbf{R}, f)$ be a Liapunov stable flow. Then a point in X is positively (or negatively) Poisson stable if and only if it is nonwandering.*

Proof. Let x be a nonwandering point in X. Then we have two sequences $\{t_n\}$ in $\mathbf{R}^+$ and $\{x_n\}$ in X such that

$$t_n \to \infty, \ x_n \to x \quad \text{and} \quad x_n t_n \to x.$$

Let $\varepsilon > 0$ be arbitrary. Then there exists $\delta > 0$ such that if $d(x, y) < \delta$ then $d(xt, yt) < \varepsilon/2$ for all $t \in \mathbf{R}^+$. Hence we have

$$\begin{aligned} d(xt_n, x) &\le d(xt_n,\ x_n t_n) + d(x_n t_n,\ x) \\ &\le \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

for sufficiently large n. This shows that the point x is positively Poisson stable.

In Example 6, the concept of Poisson stability does not imply that of recurrence. In [4] and [5], R. Knight gave some necessary conditions for a Poisson stable flow to be recurrent. In the following theorem, we get other necessary condition for a positively (or negatively) Poisson stable flow to be recurrent.

THEOREM 9. *Let $(X, \mathbf{R}, f)$ be a Liapunov stable flow on a locally compact metric space X. Then a point in X is positively (or negatively) Poisson stable if and only if it is recurrent.*

Proof. First we show taht every orbit closure $\overline{O(x)}$, $x \in X$, is minimal. Suppose that $\overline{O(x)}$ contains a nonempty closed invariant set M and let $y \in M$. For any $\varepsilon > 0$, we choose $\delta > 0$ such that if $d(y,z) < \delta$ then $d(yt, zt) < \varepsilon$ for all $t \in \mathbf{R}$. Since $y \in M \subset \overline{O(x)}$, there exists $s \in \mathbf{R}$ satisfying $d(xs, y) < \delta$. Then we have $d(x(s+t), yt) < \varepsilon$ for all $t \in \mathbf{R}$, and so $O(x) \subset B(O(y), \varepsilon) \subset B(M, \varepsilon)$. This implies that $\overline{O(x)} = M$.

Now, let x be a positively Poisson stable point in X. Then we have $\overline{O(x)} = L^+(x)$. By Theorems 12.3 and 12.8 in [1], $\overline{O(x)}$ is compact. Hence the proof is completed by Theorem 3.3.8 in [2].

It is clear that the periodicity implies the almost periodicity, but the converse does not hold even if the flow is Liapunov stable and the phase space is compact metric. Then it is interesting to study the case which the set of all periodic points in X is dense in the set of all almost periodic points in X. Here we give an example to show that the set of all periodic points in X is not dense in the set of all almost periodic points in X, even if the flow is Liapunov stable and the phase space is compact metric.

EXAMPLE 10. *Let $(T, \mathbf{R}, \pi)$ be the flow, given in Example 6, by letting $g(x,y) > 0$ for all x, y (i.e., $g(0,0) > 0$). Then every orbit is dense in the torus and moreover the torus is also positive and negative limit set of each point, and the flow is Liapunov stable. Furthermore this flow describes a compact minimal set which is not a periodic orbit and indeed each point is positively and negatively Poisson stable. Also every point in T is almost periodic by Theorems 4 and 9.*

Finally, we summarize the above results among the dynamic properties under the Liapunov stable flow as following.

THEOREM 11. *Let $(X, \mathbf{R}, f)$ be a Liapunov stable flow on a locally compact metric space X. Then the following concepts are pairwise equivalent.*

(a) almost periodic
(b) recurrent

(c) almost recurrent
(d) positively Poisson stalbe
(e) negatively Poisson stable
(f) nonwandering

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