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OPERATOR VALUED FUNCTION SPACE INTEGRALS AND NONCOMMUTATIVE OPERATIONS *

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1. Introduction

In 1968, Cameron and Storvick introduced a very general operator valued function space integral and they have established the existence of this integral as a bounded linear operator on L_2 for certain functionals [1] and set up corresponding operator–valued integrals on Yeh–Wiener space [2]. Recently, Johnson and Lapidus introduced noncommutative operations $*$ and $\dot{+}$ on Wiener functionals and could provide a precise and rigorous interpretation of certain aspects of Feynman's operational calculus for noncommuting operators [8]. In this paper, we introduce concepts of noncommutative operations $*$ and $\dot{+}$ defined on functionals on a class containing Yeh–Wiener space and study some results related to the noncommutative operations and the operator valued functional integrals.

2. Preliminaries

Let $\mathbf{C}$ denotes the complex numbers. Let $Q_{a,b} = [a,b]\times[\alpha,\beta]$. $C_2^*[Q_{a,b}] = \{x(\cdot,\cdot)\,|\,x$ is continuous on $Q_{a,b}$ and $x(\cdot,\alpha) = 0\}$ and $C_2[Q_{a,b}] = \{x(\cdot,\cdot)\,|\,x$ is continuous on $Q_{a,b}$ and $x(a,\cdot) = x(\cdot,\alpha) = 0\}$. $m_{a,b}$ will denote Yeh–Wiener measure on $C_2[Q_{a,b}]$. Frequently we will have $a = 0$ and then we will write Q_b, $C_2^*[Q_b]$, $C_2[Q_b]$ and m_b rather than $Q_{o,b}$, $C_2^*[Q_{o,b}]$, $C_2[Q_{o,b}]$ and $m_{o,b}$, respectively.

We begin by considering two restriction maps and a translation map all of which will be involved in the definition of the operations $*$ and $\dot{+}$.

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Suppose that $a < s < b$ and let $R_1 : C_2^*[Q_{a,b}] \to C_2^*[Q_{a,s}]$ be the map of restriction such that

$$[R_1(x)](u,v) = x(u,v), \ a \le u \le s, \ \alpha \le v \le \beta. \tag{2.1}$$

Similarly, let $R_2 : C_2^*[Q_{a,b}] \to C_2^*[Q_{s,b}]$ be the map of restriction such that

$$[R_2(x)](u,v) = x(u,v), \ s \le u \le b, \ \alpha \le v \le \beta. \tag{2.2}$$

Let $T : C_2^*[Q_{a,b}] \to C_2^*[Q_{b-a}]$ be the translation map,

$$[T(x)](u,v) = x(u+a,v), \ 0 \le u \le b-a, \ \alpha \le v \le \beta. \tag{2.3}$$

The restriction T_0 of T to $C_2[Q_{a,b}]$ has range $C_2[Q_{b-a}]$.

The following lemmas are well known results. We will state them without proofs [2, 6].

LEMMA 2.1.

$$m_{a,b} \circ T_0^{-1} = m_{b-a}.$$

There are three bijections P_1, P_2, P_3 onto product spaces which we will find usefulness. Given s in (a,b), $P_1 : C_2[Q_{a,b}] \to C_2[Q_{a,s}] \times C_2[Q_{s,b}]$ is defined by

$$P_1(x) = (R_1(x), \ R_2(x) - x(s,\cdot)). \tag{2.4}$$

LEMMA 2.2.

$$m_{a,b} \circ P_1^{-1} = m_{a,s} \times m_{s,b}.$$

We will often regard P_1, P_2 and P_3 as identifying the spaces involved. For example, given $x \in C_2[Q_{a,b}]$, we will often write (y,z) in place of x where $y = R_1 x$ and $z = R_2 x - x(s,\cdot)$. It is then natural to write $m_{a,b} = m_{a,s} \times m_{s,b}$ rather than the formular in Lemma 2.2.

Let $P_2 : C_2^*[Q_{a,s}] \to C_1[\alpha,\beta] \times C_2[Q_{a,s}]$ be defined by

$$P_2 x = (x(a,\cdot), \ x(\cdot,\cdot) - x(a,\cdot)). \tag{2.5}$$

We will frequently think of $C_1[\alpha,\beta] \times C_2[Q_{a,s}]$ or, under the "identification" P_2, $C_2^*[Q_{a,s}]$, as equipped with the measure $m_1 \times m_{a,s}$ where m_1 is Wiener measure on $C_1[\alpha,\beta]$.

Finally, given s with $a < s < b$, $P_3 : C_2^*[Q_{a,b}] \to C_1[\alpha,\beta] \times C_2[Q_{a,s}] \times C_2[Q_{s,b}]$ is defined by

$$P_3(x) = (x(a,\cdot),\ R_1x - x(a,\cdot),\ R_2x - x(s,\cdot)). \tag{2.6}$$

We will sometimes think of $C_1[\alpha,\beta] \times C_2[Q_{a,s}] \times C_2[Q_{s,b}]$ or, under the "identification" P_3, $C_2^*[Q_{a,b}]$, as equipped with the measure $m_1 \times m_{a,s} \times m_{s,b}$. Given (η, y, z) in $C_1[\alpha,\beta] \times C_2[Q_{a,s}] \times C_2[Q_{s,b}]$, we will often write $x = (\eta, y, z)$ rather than the more precisely correct equality $x = P_3^{-1}(\eta, y, z)$. Similarly, given (η, y) in $C_1[\alpha,\beta] \times C_2[Q_{a,s}]$, we will often write $x = (\eta, y)$ rather than $x = P_2^{-1}(\eta, y)$.

The spaces of continuous functions above are equipped with the sup norm topology. Under these topologies, R_1 and R_2 are continuous maps and T, P_1, P_2 and P_3 are all homeomorphisms.

DEFINITION 2.1. *A subset E of $C_1[\alpha,\beta]$ is said to be scale–invariant measurable provided ρE is Wiener measurable for every $\rho > 0$. A scale invariant measurable set N is said to be scale–invariant null if $m_1(\rho N) = 0$ for every $\rho > 0$. A property that holds except on a scale invariant null set is said to hold scale–invariant almost everywhere (s–a.e.).*

DEFINITION 2.2. *Let F and F_1 be $\mathbf{C}$–valued Borel measurable functions on $C_2^*[Q_{a,s}]$. We will say that F is equivalent to F_1, and write $F \sim F_1$, provided that, for every $\rho > 0$, $F(\rho x + \eta) = F_1(\rho x + \eta)$ (or $F(\eta, \rho x) = F_1(\eta, \rho x)$) for $m_1 \times m_{a,s}$ – a.e.(η, x) in $C_1[\alpha,\beta] \times C_2[Q_{a,s}]$. We will say that F is s–equivalent to F_1 and write $F \overset{s}{\sim} F_1$ if for any $\rho, \mu > 0$, $F(\rho x + \mu\eta) = F_1(\rho x + \mu\eta)$ for $m_1 \times m_{a,s}$ – a.e. (η, x) in $C_1[\alpha,\beta] \times C_2[Q_{a,s}]$. We note that is $F \overset{s}{\sim} F_1$, then $F \sim F_1$.*

PROPOSITION 2.3. *Let $a < s < b$. Suppose that $F, F_1 : C_2^*[Q_{a,s}] \to \mathbf{C}$ are Borel measurable and that $F \sim F_1$. Then $F \circ R_1$, $F_1 \circ R_1 : C_2^*[Q_{a,b}] \to \mathbf{C}$ are Borel measurable and $F \circ R_1 \sim F_1 \circ R_1$.*

Proof. The proposition is easily obtained.

PROPOSITION 2.4. *(1) Suppose* $H, H_1 : C_2^*[Q_{s,b}] \to \mathbf{C}$ *are Borel measurable functions and* $H \overset{s}{\sim} H_1$. *Then* $H \circ R_2,\ H_1 \circ R_2 : C_2^*[Q_{a,b}] \to \mathbf{C}$ *are Borel measurable and* $H \circ R_2 \sim H_1 \circ R_2$.

(2) Suppose $G,\ G_1 : C_2^*[Q_{b-s}] \to \mathbf{C}$ *are Borel measurable and that* $G \overset{s}{\sim} G_1$. *Then* $G \circ T \circ R_2$ *and* $G_1, \circ T \circ R_2$ *are Borel measurable and* $C_2^*[Q_{a,b}] \to \mathbf{C}$ *and* $G \circ T \circ R_2 \sim G_1 \circ T \circ R_2$.

Proof. (1) Clearly $H \circ R_2,\ H_1 \circ R_2$ are Borel measurable. Given $x = (\eta, y, z)$ in $C_2^*[Q_{a,b}] = C_1[\alpha,\beta] \times C_2[Q_{a,s}] \times C_2[{}_{s,b}]$, $R_2 x = R_2(\eta, y, z) = (\eta + y(s,\cdot), z) \in C_1[\alpha,\beta] \times C_2[Q_{s,b}]$. Let $\rho > 0$ be fixed. Let $M = \{(\eta, z) \in C_1[\alpha,\beta] \times C_2[Q_{s,b}] \mid H(\eta, \rho z) \neq H_1(\eta, \rho z)\}$. Since $H \sim H_1$, M is $m_1 \times m_{s,b}$–null set. Since $H \overset{s}{\sim} H_1$, for $m_{s,b}$–a.e. z $M^{(z)}$ is scale invariant null set (s–null set) where $M^{(z)}$ is z–section of M, that is, there exists $m_{s,b}$–null set N such that $M^{(z)}$ is s–null set for $z \in N^c$. Let

$$
\begin{aligned}
M'' &= \{(\xi, y, z) \in C_1[\alpha,\beta] \times C_2[Q_{a,s}] \times C_2[Q_{s,b}] \mid (H \circ R_2)(\xi, \rho y, \rho z) \\
&\qquad \neq (H_1 \circ R_2)(\xi, \rho y, \rho z)\} \\
&= \{(\xi, y, z) \mid H(\xi + \rho y(s,\cdot), \rho z) \neq H_1(\xi + \rho y(s,\cdot), \rho z)\} \\
&= \{(\xi, y, z) \mid (\xi + \rho y(s,\cdot), z) \in M\}
\end{aligned}
$$

and

$$
\begin{aligned}
M''^{(z)} &= \{(\xi, y) \mid \xi + \rho y(s,\cdot) \in M^{(z)}\} \\
&= \{(\xi, y) \mid \xi \in M^{(z)} - \rho y(s,\cdot)\} \subset [M^{(z)} - \rho y(s,\cdot)] \times C_2[Q_{a,s}].
\end{aligned}
$$

By Corollary 15 of [9], for $z \in N^c$, $M^{(z)} - \rho y(s,\cdot)$ is m_1–null with the exception of at most a s–null set of $y(s,\cdot)'s$, that is, there exists a s–null set N_1 such that $M^{(z)} - \rho y(s,\cdot)$ is m_1–null for $y(s,\cdot) \in N_1^c$. Hence

$$
\begin{aligned}
&\int_{C_1[\alpha,\beta] \times C_2[Q_{a,s}] \times C_2[Q_{s,b}]} \chi_{M''}(\xi, y, z)\, d(m_1 \times m_{a,s} \times m_{s,b})(\xi, y, z) \\
&= \int_{C_2[Q_{s,b}]/N} \int_{C_1[\alpha,\beta] \times C_2[Q_{a,s}]} \chi_{M''^{(z)}}(\xi, y)\, d(m_1 \times m_{a,s})(\xi, y)\, dm_{s,b}(z)
\end{aligned}
$$

$$\leq \int\int \chi_{[M^{(z)}-\rho y(s,\cdot)]\times C_2[Q_{a,s}]}(\xi,y)d(m_1\times m_{a,s})(\xi,y)dm_{s,b}(z)$$

$$= \int_{C_2[Q_{s,b}]/N}\int_{C_2[Q_{a,s}]}\left\{\int_{C_1[\alpha,\beta]/N_1}\chi_{[M^{(z)}-\rho y(s,\cdot)]\times C_2[Q_{a,s}]}(\xi,y)dm_1(\xi)\right.$$
$$\left. + \int_{N_1}\chi_{[M^{(z)}-\rho y(s,\cdot)]\times C_2[Q_{a,s}]}(\xi,y)dm_1(\xi)\right\} dm_{a,s}(y)dm_{s,b}(z) = 0.$$

Thus M'' is $m_1\times m_{a,s}\times m_{s,b}$–null set and so $H\circ R_2 \sim H_1\circ R_2$.

(2) Clearly $G\circ T\circ R_2$ and $G_1\circ T\circ R_2$ are Borel measurable. Let $T : C_1[\alpha,\beta]\times C_2[Q_{s,b}] \to C_1[\alpha,\beta]\times C_2[Q_{b-s}]$ be given by $T(\eta,z) = (\eta, T_0 z) = (i\circ\eta, T_0 z)$ where i is the identity map on $C_1[\alpha,\beta]$ and $T_0 : C_2[Q_{s,b}]\to C_2[Q_{b-s}]$. Hence

$$(2.7)\quad (m_1\times m_{s,b})\circ T^{-1} = (m_1\circ i^{-1})\times(m_{s,b}\circ T_0^{-1}) = m_1\times m_{b-s}.$$

Because of (1), we can establish (2) by showing that $G\circ T \overset{s}{\sim} G_1\circ T$. Let $\rho>0$, $\mu>0$ be fixed and let $M = \{(\eta,x)\in C_1[\alpha,\beta]\times C_2[Q_{b-s}]\,|\,G(\mu\eta,\rho x)\neq G_1(\mu\eta,\rho x)\}$, then M is $m_1\times m_{b-s}$–null set since $G\overset{s}{\sim}G_1$. Let $M' = \{(\eta,z)\in C_1[\alpha,\beta]\times C_2[Q_{s,b}]\,|\,(\eta,T_0z)\in M\}$. Note $M' = T^{-1}(M)$. Moreover

$$(m_1\times m_{s,b})(M') = (m_1\times m_{s,b})(T^{-1}(M)) = (m_1\times m_{b-s})(M) = 0.$$

It follows that $G(\mu\eta,\rho T_0 z) = G_1(\mu\eta,\rho T_0 z)$ for $m_1\times m_{s,b}$–a.e. (η,z) in $C_1[\alpha,\beta]\times C_2[Q_{s,b}]$ as desired.

3. Noncommutative operations $*$ and $\dot{+}$

Throughout this section, we will assume, usually without explicit mention, that all functions are Borel measurable. Also as we continue, $t, t_1, t_2, \ldots$ will denote positive real numbers.

DEFINITION 3.1. *Let* $F : C_2^*[Q_{t_1}]\to \mathbf{C}$, $G : C_2^*[Q_{t_2}]\to\mathbf{C}$. *We define* $F*G$ *and* $F\dot{+}G$ *as* $\mathbf{C}$*–valued functions on* $C_2^*[Q_{t_1+t_2}]$ *by the formulas*

$$(3.1)\qquad (F*G)(x) = F(x_1)\cdot G(x_2)$$

and

$$(F \dot{+} G)(x) = F(x_1) + G(x_2) \tag{3.2}$$

where

$$\begin{aligned} x_1(u,v) &= x(u,v),\ u \in [0,t_1],\ v \in [\alpha,\beta] \\ x_2(u,v) &= x(t_1+u,v),\ u \in [0,t_2],\ v \in [\alpha,\beta]. \end{aligned} \tag{3.3}$$

From Section 2, $x_1 = R_1 x$ and $x_2 = (T \circ R_2)(x)$. Alternatively, we can write formulas (3.1) and (3.2) as

$$(F * G)(x) = (F \circ R_1)(x) \cdot (G \circ T \circ R_2)(x) \tag{3.4}$$

$$(F \dot{+} G)(x) = (F \circ R_1)(x) + (G \circ T \circ R_2)(x). \tag{3.5}$$

THEOREM 3.1. *Let $F, F_1 : C_2^*[Q_{t_1}] \to \mathbf{C}$ and let $G, G_1 : C_2^*[Q_{t_2}] \to \mathbf{C}$. Suppose that $F \sim F_1$ and $G \overset{s}{\sim} G_1$. Then $F * G \sim F_1 * G_1$ and $F \dot{+} G \sim F_1 \dot{+} G_1$.*

Proof. Propositions 2.3 and 2.4 assure us that $F \circ R_1 \sim F_1 \circ R_1$ and $G \circ T \circ R_2 \sim G_1 \circ T \circ R_2$. The result now follows from the compatibility of $\sim$ with the usual product and sum, and from the definition of $*$ and $\dot{+}$.

THEOREM 3.2 [ALGEBRAIC PROPERTIES OF $*$].
(1) Let $F, F_1 : C_2^*[Q_{t_1}] \to \mathbf{C}$ and let $G, G_1 : C_2^*[Q_{t_2}] \to \mathbf{C}$; let $\alpha, \beta \in \mathbf{C}$. Then

$$\begin{aligned} (\alpha F + \beta F_1) * G &= \alpha(F * G) + \beta(F_1 * G) \\ F * (\alpha G + \beta G_1) &= \alpha(F * G) + \beta(F * G_1). \end{aligned} \tag{3.6}$$

(2) Let $F : C_2^*[Q_{t_1}] \to \mathbf{C}$, $G : C_2^*[Q_{t_2}] \to \mathbf{C}$ and $H : C_2^*[Q_{t_3}] \to \mathbf{C}$. Then $(F*G)*H$ and $F*(G*H)$ are $\mathbf{C}$–valued functions on $C_2^*[Q_{t_1+t_2+t_3}]$ and

$$(F * G) * H = F * (G * H). \tag{3.7}$$

(3) Let $1 : C_2^*[Q_0] \to \mathbf{C}$ *with* $1(x) = 1$ *where* $Q_0 = \{0\} \times [\alpha, \beta]$ *and* $C_2^*[Q_0] = \{x \mid x(0, v)$ *is continuous on* Q_0 *and* $x(0, \alpha) = 0\}$. *Then for all* $F : C_2^*[Q_t] \to \mathbf{C}$,

$$F * 1 = 1 * F = F. \tag{3.8}$$

Proof. The equations (3.6) and (3.8) are easily verified. The key to establish the "associativity" of $*$ is to show that, for any $x \in C_2^*[Q_{t_1+t_2+t_3}]$, both sides of (3.7), when applied to x, yield $F(x_1)G(x_2)H(x_3)$ where x_1 and x_2 are given by (3.3) and $x_3(u,v) = x(t_1 + t_2 + u, v)$, $u \in [0, t_3]$, $v \in [\alpha, \beta]$. We carry out the proof that $[(F*G)*H](x) = F(x_1)G(x_2)H(x_3)$. By definition of $*$, $[(F*G)*H](x) = (F*G)(x_1') \cdot H(x_2')$ where $x_1'(u,v) = x(u,v)$, $u \in [0, t_1 + t_2]$, $v \in [\alpha, \beta]$ and $x_2'(u,v) = x(t_1 + t_2 + u, v)$, $u \in [0, t_3]$, $v \in [\alpha, \beta]$. Note that $x_2' = x_3$ and so $H(x_2') = H(x_3)$. Hence it remains to show that $(F * G)(x_1') = F(x_1)G(x_2)$. But by definition, $(F*G)(x_1') = F(x_1'' G(x_2'')$ where $x_1''(u,v) = x_1'(u,v)$, $u \in [0, t_1]$, $v \in [\alpha, \beta]$ and $x_2''(u,v)\ x_1'(t_1 + u, v)$, $u \in [0, t_2]$, $v \in [\alpha, \beta]$. Accordingly it remains only to show that $x_1'' = x_1$ and $x_2'' = x_2$. But $t_1 + u \in [t_1, t_1 + t_2]$ and so $x_1'(t_1 + u, v) = x(t_1 + u, v)$. Therefore $x_2''(u,v) = x_1'(t_1 + u, v) = x(t_1 + u, v) = x_2(u,v)$, $u \in [0, t_2]$: that is, $x_2'' = x_2$ as desired. Finally $x_1'(u,v) = x(u,v)$ since $u \in [0, t_1]$ and so $x_1''(u,v) = x_1'(u,v) = x(u,v)$, $u \in [0, t_1]$. Thus $x_1'' = x_1$ as desired. Similarly, it can be proved that $[F * (G * H)](x) = F(x_1)G(x_2)H(x_3)$.

THEOREM 3.3 [ALGEBRAIC PROPERTIES OF $\dot{+}$].

(1) Let $F, F_1 : C_2^*[Q_{t_1}] \to \mathbf{C}$ *and let* $G, G_1 : C_2^*[Q_{t_2}] \to \mathbf{C}$; *let* $\alpha, \beta \in \mathbf{C}$. *Then*

$$(\alpha F + \beta F_1) \dot{+} (\alpha G + \beta G_1) = \alpha(F \dot{+} G) + \beta(F_1 \dot{+} G_1). \tag{3.9}$$

(2) Let $F : C_2^*[Q_{t_1}] \to \mathbf{C}$, $G : C_2^*[Q_{t_2}] \to \mathbf{C}$ *and* $H : C_2^*[Q_{t_3}] \to \mathbf{C}$. *Then*

$$(F \dot{+} G) \dot{+} H = F \dot{+} (G \dot{+} H). \tag{3.10}$$

(3) Let $0 : C_2^*[Q_0] \to \mathbf{C}$ *with* $0(x) = 0$. *Then for all* $F : C_2^*[Q_t] \to \mathbf{C}$,

$$F \dot{+} 0 = 0 \dot{+} F = F. \tag{3.11}$$

Proof. The "associativity" of $\dot{+}$ is obtained just as in the proof of (3.7) by showing that both sides of (3.10), when applied to $x \in C_2^*[Q_{t_1+t_2+t_3}]$, yield $F(x_1) + G(x_2) + H(x_3)$. As for the linearity of $\dot{+}$, we note that (3.9), when applied to $x \in C_2^*[Q_{t_1+t_2+t_3}]$, is equivalent to the following equality:

$$(\alpha F + \beta F_1)(x_1) + (\alpha G + \beta G_1)(x_2) \\ = \alpha(F(x_1) + G(x_2)) + \beta(F_1(x_1) + G_1(x_2)).$$

THEOREM 3.4. *Let $F : C_2^*[Q_{t_1}] \to \mathbf{C}$ and $G : C_2^*[Q_{t_2}] \to \mathbf{C}$.*

(1) $F\dot{+}G$, $\exp(F\dot{+}G)$ and $\exp(F) * \exp(G)$ all map $C_2^*[Q_{t_1+t_2}]$ to $\mathbf{C}$ and we have

$$\exp(F\dot{+}G) = \exp(F) * \exp(G). \tag{3.12}$$

(2) Let n be a positive integer. With the convention indicated in Remark 3.5(b) below,

$$(F\dot{+}G)^n = \sum_{p+q=n} \frac{n!}{p!q!} F^p * G^q. \tag{3.13}$$

REMARK 3.5. *(a) By (3.12) we also have $\exp(G\dot{+}F) = \exp(G) * \exp(F)$, but due to the noncommutativity involved, these quantities are not equal to $\exp(F\dot{+}G)$. An analogous comment applies to (3.13).*

(b) In (3.13) we interpret F^0 and G^0 to be 1_{t_1} and 1_{t_2}, respectively, where $1_{t_i} : C_2^*[Q_{t_i}] \to \mathbf{C}$ with $1_{t_i}(x) = 1$ for $i = 1, 2$.

Proof of Theorem 3.4. We first establish the exponential formula (3.12). For $x \in C_2^*[Q_{t_1+t_2}]$ we have

$$\begin{aligned}
[\exp(F\dot{+}G)](x) &= \exp[(F\dot{+}G)(x)] \\
&= \exp[F(x_1) + G(x_2)] \\
&= \exp[F(x_1)] \exp[G(x_2)] \\
&= (\exp F)(x_1)(\exp G)(x_2) \\
&= [(\exp F) * (\exp G)](x).
\end{aligned}$$

e next derive the binomial formula (3.13). Let $x \in C_2^*[Q_{t_1+t_2}]$.

$$(F\dot{+}G)^n(x) = [F(x_1) + G(x_2)]^n = \sum_{p+q=n} \frac{n!}{p!q!} F^p(x_1)G^q(x_2) = \sum_{p+q=n} \frac{n!}{p!q!}(F^p * G^q)(x). \tag{.14}$$

Iote that in the last equality of (3.14) we have made use of Remark ;.5(b).

4. The functional integrals K_λ^t and the operations $*$ and $\dot{+}$

DEFINITION 4.1. *Let the functional F be defined on $C_2^*[Q_{a,b}]$. Let ψ be a functional defined on $C_1[\alpha,\beta]$, let $\eta \in C_1[\alpha,\beta]$ and $\lambda > 0$. The operator valued integral $K_\lambda(F) \equiv K_{\lambda,Q_{a,b}}(F) \equiv K_\lambda^{a,b}(F)$ was defined so as to take the functional ψ into the functional $K_\lambda(F)\psi$ whose valued at η is*

$$(K_\lambda(F)\psi)(\eta(\cdot)) = \int_{C_2[Q_{a,b}]} F(\lambda^{-\frac{1}{2}}x(\cdot,\cdot) + \eta(\cdot))\psi(\lambda^{-\frac{1}{2}}x(b,\cdot) + \eta(\cdot))dx \tag{4.1}$$

where the integral is the Yeh–Wiener integral on $C_2[Q_{a,b}]$.

DEFINITION 4.2. *$W(C_1[\alpha,\beta])$ is the class of strictly Wiener measurable functional ψ defined on $C_1[\alpha,\beta]$ such that $\psi(\gamma y + \eta)$ is Wiener integrable in y over $C_1[\alpha,\beta]$ for each positive γ and each $\eta \in C_1[\alpha,\beta]$.*

It has been shown in [2] that if F is a bounded continuous functional on $C_2^*[Q_{a,b}]$, then $K_\lambda(F)$ exists as an operator on $W(C_1[\alpha,\beta])$.

THEOREM 4.1. *Let $\lambda > 0$ and suppose $F : C_2^*[Q_{t_1}] \to \mathbf{C}$ and $G : C_2^*[Q_{t_2}] \to \mathbf{C}$ are Borel measurable such that $K_\lambda^{t_1}(F) = K_{\lambda,Q_{t_1}}(F)$ and $K_\lambda^{t_2}(|G|)$ exist. Then $K_\lambda^{t_1+t_2}(F*G)$ exists and*

$$K_\lambda^{t_1+t_2}(F*G) = K_\lambda^{t_1}(F)K_\lambda^{t_2}(G). \tag{4.2}$$

Proof. Let $\lambda > 0$ be given. Let $\psi \in W(C_1[\alpha,\beta])$ and $\eta \in C_1[\alpha,\beta]$. Then

(4.3)

$$\begin{aligned}
&(K_\lambda^{t_1+t_2}(F*G)\psi)(\eta(\cdot))\\
&\overset{(1)}{=} \int_{C_2[Q_{t_1+t_2}]} (F\circ R_1)(\lambda^{-\frac{1}{2}}x(\cdot,\cdot)+\eta(\cdot))(G\circ T\circ R_2)(\lambda^{-\frac{1}{2}}x(\cdot,\cdot)+\eta(\cdot))\\
&\qquad \cdot \psi(\lambda^{-\frac{1}{2}}x(t_1+t_2,\cdot)+\eta(\cdot))dm_{t_1+t_2}(x)\\
&\overset{(2)}{=} \int_{C_2[Q_{t_1}]\times C_2[Q_{t_1,t_2+t_2}]} (F\circ R_1)(\eta(\cdot),\lambda^{-\frac{1}{2}}y,\lambda^{-\frac{1}{2}}z)\times\\
&\quad (G\circ T\circ R_2)(\eta(\cdot),\lambda^{-\frac{1}{2}}y,\lambda^{-\frac{1}{2}}z)\psi(\lambda^{-\frac{1}{2}}z(t_1+t_2,\cdot)+\lambda^{-\frac{1}{2}}y(t_1,\cdot)+\eta(\cdot))\\
&\quad \cdot d(m_{t_1}\times m_{t_1,t_1+t_2})(y,z)\\
&\overset{(3)}{=} \int_{C_2[Q_{t_1}]} F(\eta(\cdot),\lambda^{-\frac{1}{2}}y)\{\int_{C_2[Q_{t_1,t_1+t_2}]} (G\circ T)(\eta(\cdot)+\lambda^{-\frac{1}{2}}y(t_1,\cdot),\lambda^{-\frac{1}{2}}z)\\
&\qquad \cdot \psi(\lambda^{-\frac{1}{2}}z(t_1+t_2,\cdot)+\lambda^{-\frac{1}{2}}y(t_1,\cdot)+\eta(\cdot))dm_{t_1,t_1+t_2}(z)\}dm_{t_1}(y)\\
&\overset{(4)}{=} \int_{C_2[Q_{t_1}]} F(\eta(\cdot),\lambda^{-\frac{1}{2}}y)\{\int_{C_2[Q_{t_2}]} G(\lambda^{-\frac{1}{2}}w(\cdot,\cdot)+\lambda^{-\frac{1}{2}}y(t_1,\cdot)+\eta(\cdot))\\
&\qquad \cdot \psi(\lambda^{-\frac{1}{2}}w(t_2,\cdot)+\lambda^{-\frac{1}{2}}y(t_1,\cdot)+\eta(\cdot))dm_{t_2}(w)\}dm_{t_1}(y)\\
&\overset{(5)}{=} \int_{C_2[Q_{t_1}]} F(\lambda^{-\frac{1}{2}}y(\cdot,\cdot)+\eta(\cdot))\{(K_\lambda^{t_2}(G)\psi)(\lambda^{-\frac{1}{2}}y(t_1,\cdot)+\eta(\cdot))\}dm_{t_1}(y)\\
&\overset{(6)}{=}(K_\lambda^{t_1}(F)K_\lambda^{t_2}(G)\psi)(\eta(\cdot)).
\end{aligned}$$

The first equality in (4.3) above follows from the definition of $*$ and K_λ^t. The second equality follows from Lemma 2.2 : the fourth results from the application of the restriction maps given in (2.3) and from Lemma 2.1. The third equality follows from the application of the restriction maps given in (2.1) and (2.2) and from the Fubini Theorem. The use of the Fubini Theorem will be justified below. To obtain equalities (5) and (6) we have made use of the definition of K_λ^t given in (4.1). To justify the use of the Fubini Theorem, it will be convenient to work backwards beginning with expression (5) above. That expression exists with F, G

and ψ replaced by $|F|$, $|G|$ and $|\psi|$ as we next explain. Since we have assumed that $K_\lambda^{t_2}(|G|)$ exists, the function

(4.4)
$$\begin{aligned}\tilde{\psi}(\xi) &= (K_\lambda^{t_2}(|G|)|\psi|)(\xi)\\ &= \int_{C_2[Q_{t_2}]} |G(\lambda^{-\frac{1}{2}}w(\cdot,\cdot)+\xi(\cdot))|\;|\psi(\lambda^{-\frac{1}{2}}w(t_2,\cdot)+\xi(\cdot))|dm_{t_2}(w)\end{aligned}$$

belongs to $W(C_1[\alpha,\beta])$ as a function of ξ. Then, since $K_\lambda^{t_1}(F)$ existes, we know that for all $\eta \in C_1[\alpha,\beta]$,

$$\int_{C_2[Q_{t_1}]} |F(\lambda^{-\frac{1}{2}}y(\cdot,\cdot)+\eta(\cdot))|\tilde{\psi}(\lambda^{-\frac{1}{2}}y(t_1,\cdot)+\eta(\cdot))dm_{t_1}(y) < \infty. \tag{4.5}$$

By combining (4.4) and (4.5), we obtain the existence of expression (5) and (4) with absolute values on F, G and ψ. Finally, to go from (4) to (3), we use Lemma 2.1 and the change of variables theorem, but, this time, with $|F|$, $|G|$ and $|\psi|$ involved. This shows that the Fubini Theorem can be applied to (3) to yield (2).

DEFINITION 4.3. *Let $F : C_2^*[Q_{t_1}] \to \mathbf{C}$ and $G : C_2^*[Q_{t_2}] \to \mathbf{C}$. The commutator $[F,\ G]$ of F and G is a* **C***–valued function on $C_2^*[Q_{t_1+t_2}]$ which is defined by*

$$[F,G] = F*G - G*F. \tag{4.6}$$

COROLLARY 4.2. *Let $F : C_2^*[Q_{t_1}] \to \mathbf{C}$ and $G : C_2^*[Q_{t_2}] \to \mathbf{C}$ be given. Fix $\lambda > 0$. Suppose that $K_\lambda^{t_1}(|F|)$ and $K_\lambda^{t_2}(|G|)$ exist. Then $K_\lambda^{t_1+t_2}([F,G])$ exists and*

$$K_\lambda^{t_1+t_2}([F,G]) = [K_\lambda^{t_1}(F), K_\lambda^{t_2}(G)] \tag{4.7}$$

where the bracket on the right hand side of (4.7) denotes the usual commutator of linear operators on $W(C_1[\alpha,\beta])$.

Proof. The existence of $K_\lambda^{t_1+t_2}(F*G)$ and $K_\lambda^{t_1+t_2}(G*F)$ and, hence, of $K_\lambda^{t_1+t_2}([F,G])$ is insured by Theorem 4.1. A straight–forward calculation now yields (4.7) :

$$\begin{aligned} K_\lambda^{t_1+t_2}([F,G]) &= K_\lambda^{t_1+t_2}(F*G-G*F) \\ &= K_\lambda^{t_1+t_2}(F*G)-K_\lambda^{t_1+t_2}(G*F) \\ &= K_\lambda^{t_1}(F)K_\lambda^{t_2}(G)-K_\lambda^{t_2}(G)K_\lambda^{t_1}(F) \\ &= [K_\lambda^{t_1}(F),\ K_\lambda^{t_2}(G)]. \end{aligned}$$

THEOREM 4.3. *Let $F : C_2^*[Q_{t_1}] \to \mathbf{C}$ and $G : C_2^*[Q_{t_2}] \to \mathbf{C}$ be given. Fix $\lambda > 0$. Suppose that $K_\lambda^{t_1}(\exp(F))$ and $K_\lambda^{t_2}(\exp(Re\ G))$ exist. Then $K_\lambda^{t_1+t_2}(\exp(F\dot{+}G))$ exists and*

$$K_\lambda^{t_1+t_2}(\exp(F\dot{+}G)) = K_\lambda^{t_1}(\exp(F))K_\lambda^{t_2}(\exp(G)). \tag{4.8}$$

Proof. According to Theorem 4.1, we know that $K_\lambda^{t_1+t_2}(\exp(F) * \exp(G))$ exists and equals the right hand side of (4.8). Moreover, by equation (3.12) of Theorem 3.4, $\exp(F\dot{+}G) = \exp(F) * \exp(G)$. The present theorem now follows by combining these facts.

COROLLARY 4.4. *Let $F : C_2^*[Q_{t_1}] \to \mathbf{C}$ and $G : C_2^*[Q_{t_2}] \to \mathbf{C}$ be given. Fix $\lambda > 0$. Suppose that $K_\lambda^{t_1}(\exp(Re\,F))$ and $K_\lambda^{t_2}(\exp(Re\,G))$ exist. Then $K_\lambda^{t_1+t_2}(\exp(F\dot{+}G))$ and $K_\lambda^{t_1+t_2}(\exp(G\dot{+}F))$ exist and*

$$\begin{aligned} &[K_\lambda^{t_1}(\exp(F)), K_\lambda^{t_2}(\exp(G))] \\ &\qquad = K_\lambda^{t_1+t_2}(\exp(F\dot{+}G)) - K_\lambda^{t_1+t_2}(\exp(G\dot{+}F)). \end{aligned} \tag{4.9}$$

Proof. The existence of the expression on the right hand side of (4.9) is guaranted by Theorem 4.3. Further, by equation (3.12),

$$[\exp(F),\ \exp(G)] = \exp(F\dot{+}G) - \exp(G\dot{+}F). \tag{4.10}$$

Equation (4.9) now follows by applying $K_\lambda^{t_1+t_2}$ to both sides of (4.10) and using Corollary 4.2.

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