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A CHARACTERIZATION OF SUBSPACES WITH PROPERTY (I_1) IN THE ORDER NORMED SPACE OF HERMITIAN MATRICES

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1. Introduction

The ordered normed space of all $n \times n$ Hermitian matrices with the usual operator norm and with the positive cone $K = \{P \in E | x^*Px \geq 0$ for all $x \in C^n\}$, will be denoted by E or E_n. We will always assume E to be a real vector space.

If A is an arbitrary $m \times n$ complex matrix, then A^* will denote $\overline{A}^T$, i.e. the transpose of the complex conjugate of A. When $m < n$, an element $A \in E_m$ is assumed to be an element of E_n with $a_{ij} = 0$ for $i > m$ or $j > m$. Thus, E_m is considered as a subspace of E_n.

We denote E_{ij} for the $n \times n$ matrix of all zero entries except the one at (i, j) with value of 1. Note taht $E_{ij} \in E_n$ only when $i = j$. We will be using the fact that every Hermitian matrix A is diagonalizable, i.e., there exists a unitary matrix U such that U^*AU is diagonal. If D is a diagonal matrix, then we can rearrange the diagonal entries in any order that we please by a unitary transformation.

DEFINITION 1.1. *A subspace J of E is said to have Property (I_1) if for every $A \in J$ and $P \in E$ with $A,\ 0 \leq P$, there exists $Q \in J$ such that $A,\ 0 \leq Q \leq P$.*

Note that if J is a subspace with Property (I_1) and if U is a unitary matrix, then U^*JU has Property (I_1). The order Property (I_1) is equivalent to the sublattice order property in a lattice.

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DEFINITION 1.2. *A positive element P of E is said to generate an extreme ray in the positive cone K if whenever $0 \leq Q \leq P$ with $Q \in K$ then $Q = \lambda P$ for some $\lambda \geq 0$.*

LEMMA 1.3. *Let P be an element of E with $P = \begin{bmatrix} P_1 & P_3 \\ P_3^* & P_2 \end{bmatrix}$ where $P_1 \in E_m$, $P_2 \in E_{n-m}$.*
(a) If $P \geq 0$ then $P_1, P_2 \geq 0$.
(b) If $P \geq 0$ and $P_2 = 0$, then $P_3 = 0$.

Proof. Proof for part (a) is omitted. For part (b), let $x^* = (y^*, z^*)$ with $y \in C^m$, $z \in C^{n-m}$, then $x^*Px = y^*P_1y + y^*P_3z + z^*P_3^*y \geq 0$. In particular, if we take $z = \lambda P_3^* y$ with λ real, then $x^*Px = y^*P_1y + 2\lambda|P_3^*y|^2$ which is nonnegative for all $\lambda \in R$ and $y \in C^m$. Thus, we must have $P_3^*y = 0$ for all $y \in C^m$, i.e., $P_3^* = 0$.

In view of Lemma 1.3, we see that if P is positive and if all diagonal entries of P are zero, then $P = 0$.

LEMMA 1.4. *Let $D = (d_j)$ be a diagonal matrix in E where d_j is the jth diagonal entry of D and let D^+ be the diagonal matrix with $\max\{d_j, 0\}$ as the jth entry. If $D,\ 0 \leq Q \leq D^+$ for some $Q \in E$, then $Q = D^+$.*

Proof. When $d_i \geq 0$ for all $i = 1, 2, \ldots, n$ then $D = D^+$ and hence $Q = D^+$. When $d_i \leq 0$ for all $i = 1, 2, \ldots n$ then $D^+ = 0$ and therefore $Q = 0$. Thus, we assume that $d_k > 0$, $d_l < 0$ for some k and l. Since we have $0,\ d_i \leq q_{ii} \leq (D^+)_{ii}$ for all $i = 1, 2, \ldots n$, we must have $q_{ii} = (D^+)_{ii} = d_i$ in case $d_i > 0$ and $q_{ii} = (D^+)_{ii} = 0$ when $d_i < 0$. Therefore, $Q = D^+$ by Lemma 1.3.

2. A Characterization of Subspaces with Property (I_1)

LEMMA 2.1. *Let J be a subspace of E with Property (I_1) and let D be a diagonal matrix in J with $D = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}$ where $D_1 \in E_2$, $D_2 \in E_{n-2}$, and D_1 is neither positive nor negative. Assume that there exists $P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \in E$ with $P_2 = D_2^+$ and $D,\ 0 \leq P$. If both P_1 and*

$P_1 - D_1$ generate extreme rays in the positive cone of E_2, then $P \in J$.

Proof. By Property (I_1), we find $Q \in J$ with $D,\ 0 \le Q \le P$ and let $Q = \begin{bmatrix} Q_1 & Q_3 \\ Q_3^* & Q_2 \end{bmatrix}$, then we have $0,\ D_2 \le Q_2 \le P_2 = D_2^+$. Therefore, by Lemma 1.4, we have $Q_2 = D_2^+$. Now we apply Lemma 1.3 for $P - Q \ge 0$ to obtain $Q_3 = 0$. Since $0 \le Q_1 \le P_1$ and P_1 generates an extreme ray, $Q_1 = \lambda P_1$ for some $\lambda \ge 0$. Note that $\lambda \ne 0$ since Q_1 cannot be zero due to the fact that D_1 is neither positive nor negative. Similarly, we obtain $Q_1 - D_1 = \mu(P_1 - D_1)$ from $0 \le Q_1 - D_1 \le P_1 - D_1$ with $\mu > 0$. Thus, we have

$$\lambda P_1 - D_1 = Q_1 - D_1 = \mu(P_1 - D_1)$$

and hence $(\mu - \lambda)P_1 = (\mu - 1)D_1$. Now, if $\mu \ne \lambda$ then $(\mu - \lambda)P_1 \in K \cup (-K)$ and hence $(\mu-1)D_1 \in K \cup (-K)$. But D_1 is neither positive nor negative and hence $\mu - 1 = 0$, from which we obtain $Q_1 - D_1 = P_1 - D_1$, i.e., $P_1 = Q_1$. Therefore, we have $Q = P \in J$.

LEMMA 2.2. *Let J be a subspace of E with Property (I_1). If there exists a diagonal matrix $D = (d_i) \in J$ such that $d_1 > 0,\ d_2 < 0$ then $E_2 \subseteq J$.*

Proof. Let $D_0 = D - d_1E_{11} - d_2E_{22}$ then D_0 is the diagonal matrix with first two elements of value 0 and the others are identical to those of D. We define a set of positive elements of E as follows.

$$\begin{aligned}
P_0 &= D_0^+ + d_1E_{11} \\
P_1 &= D_0^+ + 2d_1E_{11} - d_2E_{22} + a(E_{12} + E_{21}), \quad a = \sqrt{-2d_1d_2} \\
P_2 &= D_0^+ + 3d_1E_{11} - 2d_2E_{22} + \sqrt{3}a(E_{12} + E_{21}), \\
P_3 &= D_0^+ + 4d_1E_{11} - 3d_2E_{22} + \sqrt{6}a(E_{12} + E_{21}), \\
Q_1 &= D_0^+ + 2d_1e_{11} - d_2E_{22} + ia(E_{12} - E_{21}), \\
Q_2 &= D_0^+ + 3d_1E_{11} - 2d_2E_{22} + i\sqrt{3}a(E_{12} - E_{21}), \\
Q_3 &= D_0^+ + 4d_1E_{11} - 3d_2E_{22} + i\sqrt{6}a(E_{12} - E_{21}).
\end{aligned}$$

we can easily check that $P_j - D_0^+ = x_jx_j^*$, $Q_i - D_0^+ = y_iy_i^*$ for $i = 1, 2, 3$

and $j = 0, 1, 2, 3$ where $x_0 = \sqrt{d_1} e_1$,

$$\begin{aligned} x_1 &= \sqrt{2d_1}\ e_1 + \sqrt{-d_2}\ e_2, \\ x_2 &= \sqrt{3d_1}\ e_1 + \sqrt{-2d_2}\ e_2, \\ x_3 &= \sqrt{4d_1}\ e_1 + \sqrt{-3d_2}\ e_2, \\ y_1 &= \sqrt{2d_1}\ e_1 - i\sqrt{-d_2}\ e_2, \\ y_2 &= \sqrt{3d_1}\ e_1 - i\sqrt{-2d_2}\ e_2, \\ y_3 &= \sqrt{4d_1}\ e_1 - i\sqrt{-3d_2}\ e_2. \end{aligned}$$

and hence $P_j - D_0^+$, $Q_i - D_0^+$ all generate extreme rays in the positive cone of E_2. Now, let

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}, \quad P_i = \begin{bmatrix} P_{i1} & 0 \\ 0 & P_{i2} \end{bmatrix}, \quad Q_i = \begin{bmatrix} Q_{i1} & 0 \\ 0 & Q_{i2} \end{bmatrix}$$

where D_1, P_{i1}, $Q_{i1} \in E_2$. Then $P_{i2} = Q_{i2} = D_0^+$ and $P_{j1} - D_1 = u_j u_j^*$, $Q_{i1} - D_1 = v_i v_i^*$ where $u_0 = \sqrt{-d_2}\ e_2$,

$$\begin{aligned} u_1 &= \sqrt{d_1}\ e_1 + \sqrt{-2d_2}\ e_2, \\ u_2 &= \sqrt{2d_1}\ e_1 + \sqrt{-3d_2}\ e_2, \\ u_3 &= \sqrt{3d_1}\ e_1 + \sqrt{-4d_2}\ e_2, \\ v_1 &= \sqrt{d_1}\ e_1 - i\sqrt{-2d_2}\ e_2, \\ v_2 &= \sqrt{2d_1}\ e_1 - i\sqrt{-3d_2}\ e_2, \\ v_3 &= \sqrt{3d_1}\ e_1 - i\sqrt{-4d_2}\ e_2. \end{aligned}$$

Hence, P_j, Q_i all satisfy the conditions described in Lemma 2.1 for $j = 0, 1, 2, 3$ and $i = 1, 2, 3$. Therefore, P_j, $Q_i \in J$. Now, note that

$$\begin{aligned} 2P_2 - P_3 - P_1 &= (2\sqrt{3} - \sqrt{6} - 1)a(E_{12} + E_{21}) \in J \\ 2Q_2 - Q_3 - Q_1 &= (2\sqrt{3} - \sqrt{6} - 1)ai(E_{12} - E_{21}) \in J \end{aligned}$$

where $a = \sqrt{-2d_1 d_2} \neq 0$ and hence $E_{12} + E_{21} \in J$, $iE_{12} - iE_{21} \in J$. Also from $2P_0 - P_1 = D_0^+ + d_2 E_{22} - a(E_{12} + E_{21}) \in J$, we have $D_0^+ + d_2 E_{22} \in J$.

If $D_0^+ = 0$, then we have $E_{22} \in J$. In case $D_0^+ \neq 0$, we apply Property (I_1) to

$$-(D_0^+ + d_2E_{22}), \quad 0 \leq -d_2E_{22}$$

to obtain $P \in J$ satisfying

$$-(D_0^+ + d_2E_{22}), \quad 0 \leq P \leq -d_2E_{22}.$$

Since E_{22} generates an extreme ray, we must have $P = \lambda E_{22}$ where $\lambda = 1$ from the above relation and hence $E_{22} \in J$. Finally, from

$$P_1 - P_0 = d_1E_{11} - d_2E_{22} - a(E_{12} + E_{21}) \in J$$

and from $E_{22} \in J$, $E_{12} + E_{21} \in J$, we obtain $E_{11} \in J$. Therefore, we conclude $E_2 \subseteq J$.

In Lemma 2.2 above, if we had $d_k > 0$ and $d_l < 0$ instead of $d_1 > 0$ and $d_2 < 0$, then we should have E_{kk}, E_{ll}, $E_{kl} + E_{lk}$, $iE_{kl} - iE_{lk} \in J$. This follows from the fact that J has Property (I_1) if and only if U^*JU has Property (I_1) for any unitary matrix U.

THEOREM 2.3. *Let J be a subspace of E with Property (I_1). If there exists a diagonal matrix $D = (d_i)$ in E_m with $d_1 > 0$, $d_2 < 0$, $d_i \neq 0$ for all $i = 3, 4, \ldots m$ such that $D \in J$ then $E_m \subseteq J$.*

Proof. By Lemma 2.2, we have $E_2 \subseteq J$ and hence if $D_0 = D - d_1E_{11} - d_2E_{22}$ then $D_0 \in J$. And for every $A \in E_2$, we have $A + D_0 \in J$. By replacing D_0 by $-D_0$ when necessary, we may assume that $d_3 < 0$. Let $D_1 = E_{22} + D_0$ then $d_2 = 1 > 0$ and $d_3 < 0$ with $D_1 \in J$. Here d_2, d_3 are the second and the third diagonal elements of D_1. From the above remark, we have $E_{33}, E_{23} + E_{32}$, $iE_{23} - iE_{32}$ are all elements of J. Now, we define $D_2 = E_{11} + D_0$ and repeat the same process to obtain $E_{13} + E_{31}$, $iE_{13} - iE_{31} \in J$. Therefore, $E_3 \subseteq J$. If $m > 3$ then we can apply the mathematical induction to finish the proof. We omit the routine proof which is a generalization of the above process.

LEMMA 2.4. *Let J be a subspace of E with Property (I_1). If $\dim J \geq 2$, then there exists $A \in J$ with $A \notin K \cup (-K)$.*

Proof. Since a subspace with Property (I_1) is positively generated, we can choose a basis $\{P_i \,|\, i = 1, 2, \ldots \alpha\}$ for J with $P_i \in K$ for all

$i = 1, 2, \dots, \alpha$. Let $D = U^*P_1U$ be diagonal with a unitary matrix U and let $Q = U^*P_2U$. We define two vectors $x = (d_i)$, $y = (q_{ii})$ where d_i and q_{ii} are the diagonal entries of D and Q respectively. Due to Lemma 1.3, neither x nor y can be a zero vector since both D and Q are positive.

First, consider the case when $\{x, y\}$ is linearly dependent, i.e., $y = \alpha x$ for some $\alpha \neq 0$. Then $Q - \alpha D \notin K \cup (-K)$ since all of the diagonal entries are zero and therefore $P_2 - \alpha P_1 = U(Q - \alpha D)U^* \in J$ with the desired property.

Now, assume that $\{x, y\}$ is linearly independent. We can multiply x, y by scalar factors so that we have $\sum_{i=1}^{\alpha} x_i = \sum_{i=1}^{\alpha} y_i = 1$. Since $\{x, y\}$ is linearly independent, $x_k \neq y_k$ for some k. If $x_k > y_k$ then we must also have $x_l < y_l$ for some l and if $x_k < y_k$ then $x_l > y_l$ for some l. Thus, $x - y \notin C \cup (-C)$ where C is the positive cone of R^n, the set of all real n–tuples. Therefore, $D - Q \notin K \cup (-K)$ where D and Q have been multiplied by same scalar factors as we did for x and y. Thus, $P_2 - P_1 = U(Q - D)U^* \notin K \cup (-K)$.

THEOREM 2.5. *Let J be a subspace of E with* $\dim J \geq 2$. *If J has Property (I_1), then there exist $1 < m \leq n$ and a unitary matrix U such that $U^*JU = E_m$.*

Proof. Since J is positively generated, we can take a basis $\{P_i \mid i = 1, 2, \dots k\}$ consisting of positive elements P_i. Let $P = \sum_{i=1}^{k} P_i$ and let $D = U^*PU$ where $D = (d_i)$ is a diagonal matrix and U is a unitary matrix. We may assume $d_i > 0$ for $i = 1, 2, \dots m$ and $d_i = 0$ for $i > m$. Since E_m is an order ideal in E due to Lemma 1.3, and since $0 \leq U^*P_iU \leq D$, we must have $U^*P_iU \in E_m$ for all $i = 1, 2, \dots k$. Thus, we have $U^*JU \subseteq E_m$.

To prove $E_m \subseteq U^*JU$, it is enough to find $A \in J$ with $A \notin K \cup (-K)$ due to Theorem 2.3. Let $Q = U^*P_1U$, $\alpha = \sum_{i=1}^{m} d_i$, $\beta = \sum_{i=1}^{m} q_{ii}$, where $q_{ii}'s$ and the diagonal entries of $Q \in E_m$. For $A = \frac{1}{\alpha}D - \frac{1}{\beta}Q$, we claim that $A \notin K \cup (-K)$. Note that if a_{ii} is the ith diagonal element of

A, then $\sum_{i=1}^{m} a_{ii} = 0$. If $a_{ii} = 0$ for all $i = 1, 2, \ldots, m$, it is clear that $A \notin K \cup (-K)$. Hence, we assume $a_{kk} > 0$ for some k. Then we also have $a_{ll} < 0$ for some l and hence $A \notin K \cup (-K)$.

LEMMA 2.6. *Let P be an element of E with $P = \begin{bmatrix} P_1 & q \\ q^* & \alpha \end{bmatrix}$ where $P_1 \in E_{n-1}$, $q \in C^{n-1}$, $0 < \alpha \in R$. Then P is positive if and only if $P_1 \geq \frac{1}{\alpha} qq^*$.*

Proof. Let $z^* = (x^*, \overline{\lambda})$ be an arbitrary n–vector with $x \in C^{n-1}$, $\lambda \in C$ and let $\omega = q^* x$. Then

$$\begin{aligned} z^* P z &= x^* P_1 x + \alpha|\lambda|^2 + \lambda x^* q + \overline{\lambda} q^* x \\ &= x^* P_1 x + \alpha|\lambda|^2 + \lambda\overline{\omega} + \overline{\lambda}\omega \\ &= x^* P_1 x + (\sqrt{\alpha}\ \lambda + \frac{1}{\sqrt{\alpha}}\ \omega)(\sqrt{\alpha}\ \overline{\lambda} + \frac{1}{\sqrt{\alpha}}\ \overline{\omega}) - \frac{1}{\alpha}|\omega|^2 \\ &= x^* P_1 x + |\sqrt{\alpha}\ \lambda + \frac{1}{\sqrt{\alpha}}\ \omega|^2 - \frac{1}{\alpha}|\omega|^2 \end{aligned}$$

Hence if $z^* P z \geq 0$ for all $z \in C^n$ then $x^* P_1 x \geq \frac{1}{\alpha}|q^* x|^2$ for all $x \in C^{n-1}$ with $\lambda = -\frac{1}{\alpha}\overline{\omega}$. Also, if $x^* P_1 x \geq \frac{1}{\alpha}|q^* x|^2$ for all $x \in C^{n-1}$, then $x^* P_1 x - \frac{1}{\alpha}|\omega|^2 + |\sqrt{\alpha}\lambda + \frac{1}{\sqrt{\alpha}}\omega|^2 \geq 0$ for all $\lambda \in C$ and $x \in C^{n-1}$. Hence $z^* P z \geq 0$ for all $z \in C^n$. Therefore, $P \geq 0$ if and only if $P_1 \geq \frac{1}{\alpha} qq^*$ since $|q^* x|^2 = (q^* x)^*(q^* x) = x^* qq^* x = x^*(qq^*)x$.

LEMMA 2.7. *Let $D = (d_i)$ be a diagonal matrix with $d_n = 0$. If D, $0 \leq P$ for some $P \in E$ then there exists $Q \in E_{n-1}$ such that D, $0 \leq Q \leq P$.*

Proof. Let $P = \begin{bmatrix} P_1 & q \\ q^* & \alpha \end{bmatrix}$ where $P_1 \in E_{n-1}$, $q \in C^{n-1}$. If $\alpha = 0$ then $q = 0$ by Lemma 1.3 and we take $Q = P$. Assume $\alpha \neq 0$ and let $Q = P_1 - \frac{1}{\alpha} qq^*$. Then by Lemma 2.6, $Q \geq 0$. Also, applying Lemma 2.6, again to $P - D \geq 0$, we obtain $(P_1 - D) - \frac{1}{\alpha} qq^* = Q - D \geq 0$. Therefore, Q satisfies D, $0 \leq Q \leq P$ and $Q \in E_{n-1}$.

THEOREM 2.8. *Let $D \in E_m$ be a diagonal matrix with $m < n$ and let $D,\ 0 \leq P$ for some $P \in E$. Then there exists $Q \in E_m$ such that $D,\ 0 \leq Q \leq P$.*

Proof. If $n - m = 1$, then the theorem follows from Lemma 2.7. When $n - m > 1$, we apply Lemma 2.7, to find $P_1 \in E_{n-1}$ such that $D,\ 0 \leq P_1 \leq P$ and apply again to $D,\ 0 \leq P_1$ to find $P_2 \in E_{n-2}$ with $D,\ 0 \leq P_2 \leq P_1$. If $n - m = 2$ then we are done since $P_2 \in E_m$ with the desired property. Otherwise, we use mathematical induction to produce $\{P_1, P_2 \cdots P_k\}$ such that $P_k \in E_{n-k}$ and $D,\ 0 \leq P_k \leq P_{k-1}$. We take $Q = P_k$ where $k = n - m$.

THEOREM 2.9. *Let J be a subspace of E with $\dim J \geq 2$. Then J has Property (I_1) if and only if there exist m with $1 < m \leq n$ and a unitary matrix U such that $U^*JU = E_m$.*

Proof. By Theorem 2.8, E_m with $1 < m \leq n$ has Property (I_1) and hence the only if part is clear. If part of this theorem is just what we had in Theorem 2.5.

It is trivial to verify that a subspace J with $\dim J = 1$ has Property (I_1) if and only if $J = \{\lambda P | \lambda \in R\}$ for some $P \in K$. Hence, Theorem 2.9 provides a complete characterization of a subspace with Property (I_1). We also note that a subspace with Property (I_1) must have dimension k^2 for some positive integer k since $\dim(E_k) = k^2$.

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