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CONVERGENCE OF GALERKIN APPROXIMATION FOR A HEAT EQUATION WITH A RANDOM INITIAL CONDITION *

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1. Introduction

The purpose of this paper is to present some analyses of probabilistic convergence of Galerkin approximations for a heat equation with a random initial condition which has been studied in [7, 8] by S. Tasaka.

We consider the problem:

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \text{ in } D \times (0,T] \times \Omega \\ &u(x,0,\omega) = u_0(t,\omega) \text{ at } t = 0 \\ &u(x,t,\omega) = 0 \text{ in } \partial D \times [0,T] \times \Omega, \end{aligned} \tag{1.1}$$

where $D \equiv (0,1) \subset \mathrm{R}$ with boundary $\partial D, 0 < T < \infty, (\Omega, \Sigma, P)$ is a complete probability space and $u_0(t,\omega)$ a random initial condition on (Ω, Σ, P). For each $\omega \in \Omega$, the problem (1.1) is an ordinary heat equation. If for each $\omega \in \Omega$, we have the explicit formula for the sample path $u_0(\cdot,\omega)$ then one can solve the equation (1.1) for $u(\cdot,\omega)$ for each w in the ordinary way. However in general one has information only about the statistical data of $u(x,0,\omega)$, i.e. the mean and variance etc.; one does not know the sample path $u_0(x,0,\omega)$ explicitly. In this case one can get random approximate solutions and investigate their probabilistic convergent properties. The methods followed here are similar to the ones in [2] [4], [6] and [7] except repeated application of the Markov's inequality and the Borel-Cantelli's lemma.

2. Notations

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Let (Ω, Σ, P) be independent of time t. We denote by $L^2(0,1) = L^2, H^m(0,1) = H^m$ and $H_0^m(0,1) = H_0^m, m = 1, 2, \cdots$, the usual Lebesgue and Sobolev spaces on (0,1) respectively and by $(\cdot,\cdot)$, $\|\cdot\|_0$ and $\|\cdot\|_m$ their natural inner product and norms respectively. For a sequence of discretization parameter $h_j \in (0, \frac{1}{2}], j = 1, 2, \cdots$, with $h_j \downarrow 0$, let $S^{h_j} \subset H_0^1$ be finite dimensional subspaces such that for all $v \in H_0^1 \cap H^q, q \in \{1, 2\}$,

$$\inf \|v - \phi^{h_j}\|_p \leq C h_j^{q-p} \|v\|_q,\ p \in \{0,1\},\ \phi^{h_j} \in S^{h_j},\ p < q, \tag{2.1}$$

where the constant C does not depend on h or v.

3. Almost surely (a.s.) convergence of continuous time Galerkin approximations

The variational formulation of (1.1) is

$$\begin{aligned} &(u_t(t,\omega), \phi) + (\nabla u(t,\omega), \nabla\phi) = 0,\ \forall\, \phi \in H_0^1 \text{ a.s. and} \\ &u(0,\omega) = u_0(\omega) \text{ a.s.}\,. \end{aligned} \tag{3.1}$$

where $u_t = \frac{\partial u}{\partial t}$ and $\nabla u = \frac{\partial u}{\partial x}$.

DEFINITION. A random function $u : D \times [0,T] \times \Omega \to \mathrm{R}$ is a random weak solution of (1.1) iff

(i) $u(x,t,\omega)$ is measurable for every $(x,t) \in D \times [0,T]$
(ii) $u(\cdot,\cdot,\omega) \in C^1([0,T] : H_0^1)$ for a.s. and
(iii) $u(x,t,\omega)$ satisfies (3.1) a.s.

For each $\omega \in \Omega$, (3.1) is a deterministic problem which has a unique weak solution $u(\omega) : [0,T] \to H_0^1$ [3]. By the same reason there exists a unique continuous time Galerkin approximation $u^{h_j}(\omega) : [0,T] \to S^{h_j}$ to $u(\omega)$ defined by

$$\begin{aligned} &(u_t^{h_j}(t,\omega), \phi^{h_j}) + (\nabla u^{h_j}(t,\omega), \nabla\phi^{h_j}) = 0\ \forall \phi^{h_j} \in S^{h_j},\ t > 0, \\ &u^{h_j}(0) = L^{h_j} u_0(\omega),\ \omega \in \Omega, \end{aligned} \tag{3.2}$$

where $L^{h_j} u_0(\omega)$ is the L^2-projection of $u_0(\omega)$ into S^{h_j}. One way to show the measurability of the solution (3.1) is to make use of the Fadeo-Galerkin procedure in (3.2), but it is lengthy. So it is omitted. See [3]. We shall use the Ritz projection R^{h_j} defined by

$$(\nabla R^{h_j} u(t,\omega), \nabla\phi^{h_j}) = (\nabla u(t,\omega), \nabla\phi^{h_j}),\ \forall \phi^{h_j} \in S^{h_j} \tag{3.3}$$

Note that R^{h_j} commutes with time derivative. We write

$$\begin{aligned} e^{h_j}(t,\omega) &= u^{h_j}(t,\omega) - u(t,\omega) \\ &= u^{h_j}(t,\omega) - R^{h_j}u(t,\omega) + R^{h_j}u(t,\omega) - u(t,\omega) \\ &= \theta^{h_j}(t,\omega) + \rho^{h_j}(t,\omega), \end{aligned}$$

where

$$\theta^{h_j}(t,\omega) = u^{h_j}(t,\omega) - R^{h_j}u(t,\omega)$$

and

$$\rho^{h_j}(t,\omega) = R^{h_j}u(t,\omega) - u(t,\omega).$$

Throughout, $C > 0$ is a generic constant and

$$\langle v(t)\rangle = \int_\Omega v(t,\omega)\, dP(\omega).$$

THEOREM 3.1. *Suppose* $\sum_{j=1}^{\infty} h_j^{\beta-p} < \infty$, $p \in (0,1], \beta \in \{1,2\}, p < \beta$ *and* $u_0(\omega) \in H_0^1 \cap H^\beta$ *a.s. . If* $\left\langle \|u_0\|_\beta^2 \right\rangle < \infty$ *and* $\left\langle \|u_t\|_\beta^2 \right\rangle < \infty$, *then* $\|u^{h_j}(t,\omega) - u(t,\omega)\|_0$ *converges a.s. to zero with the rate* $O(h_j^p)$.

PROOF: Observe first that $\|v - R^{h_j}v\|_p \le Ch_j^{\beta-p}\|v\|_\beta, v \in H_0^1 \cap H^\beta$. Differentiate (3.3) with respect to t and use the above approximability property to obtain

$$\|\theta_t^{h_j}(t,\omega)\|_0 \le ch_j^\beta\big\{\|u_0(\omega)\|_\beta + \int_0^T \|u_s(\omega)\|_\beta ds\big\}. \tag{3.4}$$

It follow then that for each $\omega \in \Omega$,

$$\|e^{h_j}(t,\omega)\|_0 \le Ch_j^\beta\big\{\|u_0(\omega)\|_\beta + \|u(t,\omega)\|_\beta + \int_0^T \|u_s(\omega)\|_\beta\, ds\big\}. \tag{3.5}$$

Thus taking expectation on both sides of (3.5) yields

$$\begin{aligned} \langle\|e^{h_j}(t)\|_0^2\rangle \le Ch_j^{2\beta}\big\{\langle\|u_0\|_\beta^2\rangle + \langle\|u(t)\|_\beta^2\rangle \\ + T\int_0^T \langle\|u_s\|_\beta^2\rangle\, ds\big\} \le Ch_j^{2\beta}. \end{aligned} \tag{3.6}$$

According to the Markov's inequality and (3.6), we have

$$P(\|e^{h_j}(t,\omega)\|_0 \geq h_j^p) \leq \frac{\langle \|e^{h_j}(t)\|_0^2 \rangle}{h_j^{2p}} \leq C h_j^{2(\beta-p)}, \tag{3.7}$$

which leads by the hypothesis to

$$\sum_{j=1}^{\infty} P(\|e^{h_j}(t,\omega)\|_0 \geq h_j^p) \leq C \sum_{j=1}^{\infty} h_j^{2(\beta-p)} < \infty. \tag{3.8}$$

Now let A_{h_j} and B be the events

$$A_{h_j} = \{w : \|e^{h_j}(t,\omega)\|_0 \geq h_j^p\}$$

and

$$B = \{\omega : \|e^{h_j}(t,\omega)\|_0 \geq h_j^p, \text{ infinitely often}\}.$$

Note that $B = \lim_{j\to\infty} \sup A_{h_j}$. Thus finally the Borel-Cantelli's lemma to (3.7) yields $P(B) = 0$ which implies the assertion.

Let $\{h_{j_\alpha}\}_{\alpha=1}^N$ be a random finite subsequence of $\{h_j\}_{j=1}^\infty$ with $\sum_{j=1}^\infty h_j^{\beta-p} < \infty, p \in \{0,1\}, \beta \in \{1,2\}$ and h_{j_α}'s are not necessary different but not infinitely many same h_{j_α}'s when $N \to \infty$. We call $\{h_{j_\alpha}\}_{\alpha=1}^N$ the sample meshes of size N from $\{h_j\}_{j=1}^\infty$.

DEFINITION. Define the Galerkin sample mean $\bar{u}_N$ by

$$\bar{u}_N(t,\omega) = \frac{1}{N} \sum_{a=1}^{N} u^{h_{j\alpha}}(t,\omega), \tag{3.9}$$

where $u^{h_{j\alpha}}(t,\omega)$ is the solution of (3.2).

THEOREM 3.2. *Let $p \in \{0,1\}$, $\beta \in \{1,2\}$ and $p < \beta$, and $\{h_{j_\alpha}\}_{\alpha=1}^N$ be a random finite subsequence as above. If $\left\langle \|u_0\|_\beta^2 \right\rangle$, $\left\langle \|u(t)\|_\beta^2 \right\rangle$ and $\left\langle \|u_t\|_\beta^2 \right\rangle$ are finite, then the Galerkin sample mean $\bar{u}_N$ converges in probability to $\langle u(t) \rangle$ in $\|\cdot\|_p$-norm.*

PROOF: We write

$$
\begin{aligned}
&\|\bar{u}_N(t,\omega) - \langle u(t)\rangle\|_p^2 \\
=&\frac{1}{N^2}\Big\| \sum_{\alpha=1}^{N} \{(u^{h_{h\alpha}}(t,\omega) - R^{h_{j\alpha}}u(t,\omega)) + (R^{h_{j\alpha}}u(t,\omega) - u(t,\omega)) \\
&\qquad + (u(t,\omega) - \langle u(t)\rangle)\}\Big\|_p^2 \\
\leq&\frac{1}{N}\Big\{\sum_{\alpha=1}^{N}[\|\theta^{h_{j\alpha}}(t,\omega)\|_p^2 + \|\rho^{h_{j\alpha}}(t,\omega)\|_p^2] + \|u(t,\omega) - \langle u(t)\rangle\|_p^2\Big\},
\end{aligned}
$$

and take the expectation to get

$$
\begin{aligned}
(3.10)\qquad &\langle\|\bar{u}_N(t) - \langle u(t)\rangle\|_p^2\rangle \\
&\leq \frac{1}{N}\{\langle\|u_0\|_\beta^2\rangle + \langle\|u(t)\|_\beta^2\rangle + T\int_0^T \langle\|u_t\|_\beta^2\rangle\, dt\} \\
&\quad\times\sum_{j=1}^{\infty} h_j^{2(\beta-p)} \to 0 \quad \text{as} \quad N\to\infty,
\end{aligned}
$$

which completes the proof.

COROLLARY. *Let N_j's be the sample size corresponding to h_j's with $\sum_{j=1}^{\infty} h_j^{\beta-\alpha}<\infty$, $0<\alpha<\beta$. If $\sum_{j=1}^{\infty} N_j^{-1}h_j^{-2\alpha}<\infty$, then $\|\bar{u}_N(t,\omega) - \langle u(t)\rangle\|_p = 0(h_j^\alpha)$ a.s. .*

PROOF: We start with

$$
\begin{aligned}
&\sum_{j=1}^{\infty} P(\|\bar{u}_N(t,\omega) - \langle u(t)\rangle\|_p \geq h_j^\alpha) \\
&\leq \sum_{j=1}^{\infty}\frac{\langle\|\bar{u}_N(t) - \langle u(t)\rangle\|_p^2\rangle}{h_j^{2\alpha}} \\
&\leq C\sum_{j=1}^{\infty} N_j^{-1}h_j^{-2\alpha} < \infty,
\end{aligned}
$$

which leads by the previous arguments to the result.

REMARK 1. Given a sequence of meshes $\{h_j\}_{j=1}^{\infty}$ with $\sum_{j=1^\infty} h_j^{\beta-p} < \infty$, $p \in \{0,1\}$, $\beta \in \{1,2\}$, $p < \beta$, one can choose N_j by $N_j \geq [h_j^{-2p}] + 1$, where $[x]$ denotes the largest integer less than or equal to x. Thus by the corollary one can expect that the Galerkin sample mean will be close to the expectation $\langle u(t)\rangle$ of the exact solution u in an appropriate norm with a high probability as much as desired, that is, for every $\epsilon > 0$ and $\delta > 0$, there exists a number N such that

$$P\left(\max_{N>N_0} \|\bar{u}_N(t,\omega) - \langle u(t)\rangle\|_p < \epsilon\right) \geq 1 - \delta.$$

It is only by the strong law of large numbers that the existence of such N is indeed guaranteed. This means that the continuous time Galerkin sample mean is better estimator for $\langle u(t)\rangle$ than the solution u_A of the averaged equation when $\langle u(t)\rangle \neq u_A$ which occurs in many general situations.

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