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AN ALGORITHM FOR BIVARIATE GAUSSIAN QUADRATURE

HA-JINE KIMN

1. Introduction

To evaluate an univariate definite integral by numerical method, various methods can be available, for example, trapezoidal rule, Simpson's rule and Gaussian quadrature, etc. Generally Simpson's rule requires much more computing time, since it needs more function evaluations than those of Gaussian quadrature. One reason why there are many formulas for numerical integration is that there are so many ways for selecting the space of base points and the degree of interpolating polynomials. These formulas can be classified into two groups : those of equally spaced points and those for which the spacing is unequal but not arbitrary. Gaussian quadrature formulas are the latter. The numerical integration with equally spaced base points is valid only under the condition that interval $[a, b]$ is finite and the integrand $f(x)$ is sufficiently differenciable.

Univariate Gaussian quadrature has been developed by many authors. [2], [3], [4] However, bivariate quadrature, specially, bivariate Gaussian quadrature has not yet been clearly searched. The bivariate quadrature is strongly demended for various numerical computations.

The object of this note is to establish an algorithm for bivariate Gaussian quadrature by dealing with an expansion of the univariate definite integral to the bivariate integral of general form :

$$\int_a^b \int_{c(x)}^{d(x)} f(x,y)dy\,dx.$$

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2. On the univariate Gaussian quadrature

In the case of non equally spaced points, the definite integral can be written as

$$\int_a^b f(x)dx = \int_a^b g(x)w(x)dx \approx A_0 g(x_0) + \cdots + A_k g(x_k) \tag{2.1}$$

where $A_0, \cdots, A_k$ do not depend upon $g(x)$. [4] We need to calculate these $A_0, \cdots, A_k$. Since Formula (2.1) has $2k+2$ parameters $x_0, \cdots, x_k$, $A_0, \cdots, A_k$, it is helpful to derive a formula which is exact for all polynomials of degree $2k+1$. [1]

Let $f(x) = g(x)w(x)$ and $g(x)$ be an approximation of $P_k(x)$ which is a polynomial of degree not exceeding k and which interpolates $g(x)$ at $x_0, \cdots, x_k$ on (a, b). Then we have the Newton form

$$g(x) = P_k(x) + g[x_0, \cdots, x_k, x] \cdot (x - x_0) \cdots (x - x_k)$$

where $g[x_0, \cdots, x_k, x]$ is the $(k+1)$th divided difference of $g(x)$. Integrating $f(x)$ on $[a, b]$,

$$\begin{aligned}\int_a^b f(x)dx &= \int_a^b g(x)w(x)dx \\ &= \int_a^b P_k(x)w(x)dx \\ &\quad + \int_a^b g[x_0, \cdots, x_k, x] \cdot (x - x_0) \cdots (x - x_k)w(x)dx.\end{aligned}$$

By Lagrange form, $P_k(x)$ is to be of the form

$$P_k(x) = g(x_0)l_0(x) + \cdots + g(x_k)l_k(x)$$

where $l_i(x)$ $(i = 0, \cdots, k)$ is the $(i+1)$th Lagrange basis. Thus

$$\begin{aligned}&\int_a^b P_k(x)w(x)dx \\ &= g(x_0)\int_a^b l_0(x)w(x)dx + \cdots + g(x_k)\int_a^b l_k(x)w(x)dx \\ &= A_0 g(x_0) + \cdots + A_k g(x_k).\end{aligned} \tag{2.2}$$

where $A_i = \int_a^b l_i(x)w(x)dx$.

To estimate the error term of integration, we consider :

$$\int_a^b g(x)w(x)dx - \int_a^b P_k(x)w(x)dx = \int_a^b g[x_0, \cdots, x_k, x]\psi_k(x)w(x)dx \tag{2.3}$$

with $\psi_k(x) = (x - x_0)\cdots(x - x_k)$. Let $P_{k+1}(x)$ be an orthogonal polynomial of degree $k+1$ with respect to $w(x)$ on (a, b). Then by the properties of orthogonal polynomial, we have

$$P_{k+1}(x) = \alpha(x - \xi_0)(x - \xi_1)\cdots(x - \xi_k)$$

with the $(k+1)$ distinct points $\xi_0, \cdots, \xi_k$ in (a, b) and a constant α, and

$$\int_a^b P_{k+1}(x)w(x)dx = 0.$$

Thus, from (2.3), we have the error term :

$$\begin{aligned} &\int_a^b g\big[x_0, \cdots, x_{2k+1}, x\big]\psi_{2k+1}(x)w(x)dx \\ &= \int_a^b g\big[x_0, \cdots, x_{2k+1}, x\big]\psi_k(x)(x - x_{k+1})\cdots(x - x_{2k+1})w(x)dx \\ &= \frac{g^{(2k+2)}(\zeta)}{(2k+2)!}\int_a^b \frac{1}{\alpha^2}((x - \xi_0)\cdots(x - \xi_k))^2 w(x)dx, \qquad \zeta \in (a, b). \end{aligned} \tag{2.4}$$

by the differentiation property of divided difference.

For the concrete computation of (2.1), let $w(x)$ be a positive integrable function on (a, b), and let $\{P_0(x), \cdots, P_{k+1}(x)\}$ be a sequence of orthogonal polynomals with respect to $w(x)$ on (a, b). Furthermore, suppose that $P_{k+1}(x)$ has $k+1$ distinct real roots $\xi_0, \cdots, \xi_k$ in (a, b). Then

$$\int_a^b f(x)dx = \int_a^b g(x)w(x)dx \approx \sum_{i=0}^{k} A_i g(\xi_i) \tag{2.5}$$

where

$$A_i = \int_a^b \prod_{\substack{j=0 \\ j\neq i}}^{k} \frac{(x-\xi_j)}{(x_i-\xi_j)} w(x)dx.$$

Equation (2.4) shows that if we substitue any polynomial of degree not exceeding $(2k+1)$ for $g(x)$ of (2.5), then $g^{(2k+2)}(\zeta)$ becomes zero. The above formula (2.5) is referred to as *n–point univariate Gaussian quadrature formula* and the ξ_k's are called *Gaussian base points on* $[a,b]$, and A_i's their *Gaussian weights.*

Various Gaussian type quadrature rule can be derived according to their orthogonal polynomial, for example, *Gauss–Legendre, Gauss–Chebyshev, Gauss–Laguerre* and *Gauss–Hermite quadrature rule.* Especially, we know that the Legendre polynomial $L_i(x)$ is orthogonal with respect to $w(x)=1$ on $[-1,1]$. Whose recurrence relation is

$$L_{i+1}(t) = \frac{2i+1}{i+1} t \cdot L_i(t) - \frac{i}{i+1} L_{i-1}(t)$$

with $L_0(t)=1$ and $L_1(t)=t$.

By the linear change of variable:

$$t = \frac{2x-(a+b)}{b-a},$$

our definite integral becomes as follow:

$$\int_a^b f(x)dx = \int_{-1}^1 F(t)dt$$

with $F(t) = f(x(t))x'(t)$. Thus the $k+1$ points Gauss–Legendre formula is

$$\int_a^b f(x)dx = \int_{-1}^1 F(t)dt = \sum_{i=1}^{k} A_i F(t_i) \tag{2.6}$$

where $t_0, \cdots, t_k$ are the zeros of Legendre polynomial of degree $k+1$ and

$$A_i = \int_{-1}^1 \prod_{\substack{j=0 \\ j\neq i}}^{k} \frac{(t-t_j)}{(t_i-t_j)} dt.$$

In general, Gauss-Legendre quardrature (2.6) is called the $(k+1)$ *points Gauss quadrature.*

3. An algorithm for bivariate Gaussian quadrature

An expansion of above univariate Gaussian quadrature can make an algorithm for bivariate Gaussian quadrature as follow.

The most general form of bivariate integral is

$$I = \int_a^b \int_{c(x)}^{d(x)} f(x,y)dy\,dx.$$

Let x_i $(i = 1, \cdots, N)$ and y_k $(k = 1, \cdots, M)$ be the base points with respect to x–coordinates and y–coordinates respectively, and let w_i $(i = 1, \cdots, N)$ and w_k $(k = 1, \cdots, M)$ be the corresponding weights.

1^0) We evaluate N inner integrals with respect to x–coordinates.

$$g(x_1) = \int_{c(x_1)}^{d(x_1)} f(x_1, y)dy.$$

$$\cdots\cdots$$

$$g(x_N) = \int_{c(x_N)}^{d(x_N)} f(x_N, y)dy.$$

Thus

$$I \approx \frac{b-a}{2} \sum_{i=1}^{N} g(x_i)w_i \tag{3.1}$$

where $g(x_i) = \int_{c(x_j)}^{d(x_i)} f(x_i, y)dy.$

2^0) We evaluate $g(x_i)$ using y_k and its weight $w_{i,k}$ $(k = 1, \cdots, M)$. That is,

$$g(x_i) = \frac{d(x_i) - c(x_i)}{2} \sum_{k=1}^{M} f(x_i, y_k)w_{i,k} \tag{3.2}$$

Substituting (3.2) into (3.1), we have

$$I = \int_a^b \int_{c(x)}^{d(x)} f(x,y)dy\,dx \approx \frac{b-a}{2}\sum_{i=1}^{N}\{\frac{d(x_i)-c(x_i)}{2}\sum_{k=1}^{M} f(x_i,y_k)w_{i,k}\}w_i. \tag{3.3}$$

REMARKS. The proof of (3.3) is not necessary because of its clearness. The error analysis of (3.3) will be worked soon or late.

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Ajou University
Suwon 440–749, Korea