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ON THE CONSTRUCTION OF QUATERNION FIELDS *

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Let N be a normal algebraic number field whose Galois group $Gal(N/Q)$ over the field Q of rational numbers is the quaternion group of order 8. Such a number field is called quaternion field. In this paper we shall construct all quaternion fields of discriminant $\leq 10^{16}$. There are exactly 33 totally real fields and 37 totally imaginary fields.

1. Discriminant

If F is a number field, d_F denotes the discriminant of the extension F/Q and $N_{F/Q}$ denotes the norm from F to Q. Let p be a prime number. If p^n divides d_F and if p^{n+1} does not divide d_F; the integer n is called the exponent of p in d_F and is denoted $v_p(d_F)$.

Let N be a quaternion field, K the biquadratic subfield of N and let k_1, k_2, k_3 be three quadratic subfields of K.

PROPOSITION 1. *Let p be an odd prime number, ramified in the extension N/Q. Then*

$$v_p(d_N) = 6 \text{ if } p \text{ is ramified in the extension } K/Q$$
$$v_p(d_N) = 4 \text{ if not.}$$

Proof. Let us denote by $\delta_{N/K}$ the discriminant of the extension N/K. If p is an odd prime number, ramified in N/Q, then p is tamely ramified. The exponent of p in d_N follows immediately from the fact that

$$d_N = d_K^2 \cdot N_{K/Q}(\delta_{N/K}) \quad \text{(cf. Prop. 8 Ch.III[S1])}$$

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Write $N = K(\sqrt{\alpha})$ for square $\alpha \in K$. We shall say that 2 is partially ramified in the extension K/Q if 2 is ramified in K/Q and if 2 is not totally ramified in K/Q.

PROPOSITION 2.

i) *If 2 is totally ramified in* K/Q, *then* $v_2(d_N) = 24$.

ii) *If 2 is partially ramified in* K/Q, *then* $v_2(d_N) = 22$.

iii) *When 2 is not ramified in* K/Q, *we choose* α *an integral over* Q, *not divisible by 4 in* K *such that* $N = K(\sqrt{\alpha})$. *Then we have :*

$$\begin{aligned} v_2(d_N) &= 12 \quad \text{if 2 divides } \alpha, \\ v_2(d_N) &= \ \ 8 \quad \text{if 2 does not divide } \alpha \text{ and} \\ &\qquad\ \ \text{if the congruence } \alpha \equiv x^2 \text{ can not be solved in } K, \\ \text{and} \quad v_2(d_N) &= \ \ 0 \quad \text{if 2 does not divide } \alpha \text{ and} \\ &\qquad\ \ \text{if the congruence } \alpha \equiv x^2 \text{ can be solved in } K. \end{aligned}$$

Proof. In prop. 1. [M1], the jumps in the filtration of ramification groups are determined. i) and ii) follow from prop.4 Ch.IV.[S1]. We can easily deduce iii) from the Kummer theory (cf. Th. 119 [H]).

2. Construction of Quaternion Fields

PROPOSITION 3. *Let* K *be a biquadratic field. Write* $K = Q(\sqrt{m_1}, \sqrt{m_2})$, $k_i = Q(\sqrt{m_i})$ *with* m_i *square free integers. The field* K *is a subfield of a quaternion field if and only if*

$$(-1, m_1)_p \cdot (-1, m_2)_p \cdot (m_1, m_2)_p = 1$$

for all rational places p *(including the infinite prime).*

A proof of this proposition is given in [F].

By Th.1. Ch.III [S2], we obtain easily

PROPOSITION 4. *A necessary and sufficient condition for* $K = Q(\sqrt{m_1}, \sqrt{m_2})$ *to be a subfield of a quaternion field is that the following conditions hold :*

i) m_1 and m_2 are > 0,

ii) *for every odd prime divisor p of d_K, we have*

$$(\frac{-1}{p}) = (\frac{m_i}{p})$$

where i is such that p is not ramified in the extension $Q(\sqrt{m_i})/Q$.

iii) *if 2 is partially ramified in K/Q, then*

$$m_1 \equiv 1 \ (mod\ 8) \text{ and } m_2 \equiv 2 \ (mod\ 8)$$

or

$$m_1 \equiv 5 \ (mod\ 8) \text{ and } m_2 \equiv -2 \ (mod\ 8).$$

iv) *if 2 is totally ramified in K/Q, then*

$$m_1 \equiv 3 \ (mod\ 8) \text{ and } m_2 \equiv \pm 2 \ (mod\ 8).$$

Let K be a biquadratic field satisfying the conditions in the proposition 4. Let α be an element of K. The field $K(\sqrt{\alpha})$ is a quaternion field if and only if $N_{K/k_i}(\alpha)$ are of the form $m_j\lambda_i^2$, with $j \neq i$ and $\lambda_i \in k_i$. In this case, $K(\sqrt{\alpha})$ is a normal extension over Q and $K(\sqrt{\alpha})$ is cyclic over k_1, k_2 and k_3 respectively.

We determine the biquadratic fields K which can be imbedded in a quaternion field of discriminant $\leq 10^{16}$. Using the proposition 4 we find at first the fields K such that $d_K^2 \leq 10^{16}$. Then we compute the discriminant d_N of a pure quaternion field. If $K(\sqrt{M})$ denotes a pure quaternion field, then every quaternion field containing K is of the form $N_m = K(\sqrt{mM})$ for some integer m (cf. 2 [M2]).

Using proposition 1 and 2 we find finally all fields N_m of discriminant $\leq 10^{16}$. At the end of this paper we will give the lists of all quaternion fields of discriminant $\leq 10^{16}$,

Table 1 : all fields N with $v_2(d_N) = 24$

Table 2 : all fields N with $v_2(d_N) = 22$

Table 3 : all fields N with $v_2(d_N) = 0.8$ or 12

In particular, we prove

PROPOSITION 5. *The smallest discriminant of quaternion field is* $2^{24}\cdot 3^6 = 12,230,590,464$. *There are, up to isomorphism, one real field and one imaginary field ; they are the fields* $Q\left(\sqrt{\pm(2+\sqrt{2})(3+\sqrt{3})}\right)$.

Table 1

K	$\sqrt{d_k}$	M	α	d_N
$Q(\sqrt{2},\sqrt{3})$	48	$(2+\sqrt{2})(3+\sqrt{3})$	$\pm M$	$2^{24}\cdot 3^6(\doteqdot 1.223\times 10^{10})$
			$\pm 5M$	$2^{24}\cdot 3^6\cdot 5^4(\doteqdot 7.644\times 10^{12})$
			$\pm 7M$	$2^{24}\cdot 3^6\cdot 7^4(\doteqdot 2.936\times 10^{13})$
			$\pm 11M$	$2^{24}\cdot 3^6\cdot 11^4(\doteqdot 1.790\times 10^{14})$
			$\pm 13M$	$2^{24}\cdot 3^6\cdot 13^4(\doteqdot 3.493\times 10^{14})$
			$\pm 17M$	$2^{24}\cdot 3^6\cdot 17^4(\doteqdot 1.021\times 10^{15})$
			$\pm 19M$	$2^{24}\cdot 3^6\cdot 19^4(\doteqdot 1.593\times 10^{15})$
			$\pm 23M$	$2^{24}\cdot 3^6\cdot 23^4(\doteqdot 3.422\times 10^{15})$
			$\pm 29M$	$2^{24}\cdot 3^6\cdot 29^4(\doteqdot 8.650\times 10^{15})$
$Q(\sqrt{3},\sqrt{14})$	336	$(3+\sqrt{3})(4+\sqrt{14})(7+\sqrt{42})$	$\pm M$	$2^{24}\cdot 3^6\cdot 7^6(\doteqdot 1.438\times 10^{1}5)$
			$\pm 2M$	$2^{24}\cdot 3^6\cdot 7^6(\doteqdot 1.438\times 10^{15})$
$Q(\sqrt{2},\sqrt{12})$	176	$(2+\sqrt{2})(11+3\sqrt{11})$	$\pm M$	$2^{24}\cdot 11^6(\doteqdot 2.972\times 10^{13})$
			$\pm 3M$	$2^{24}\cdot 11^6\cdot 3^4(\doteqdot 2.407\times 10^{15})$
$Q(\sqrt{2},\sqrt{19})$	304	$(2+\sqrt{2})(19+\sqrt{19})$	$\pm M$	$2^{24}\cdot 19^6(\doteqdot 7.892\times 10^{14})$

Table 2

K	$\sqrt{d_k}$	M	α	d_N
$Q(\sqrt{5},\sqrt{6})$	120	$(5+\sqrt{5})(6+\sqrt{6})$	$\pm M$	$2^{22}\cdot 5^6\cdot 3^6(\doteqdot 4.777\times 10^{13})$
			$\pm 2M$	$2^{22}\cdot 5^6\cdot 3^6(\doteqdot 4.777\times 10^{13})$
$Q(\sqrt{5},\sqrt{14})$	280	$(5+\sqrt{5})(14+3\sqrt{14})$	$\pm M$	$2^{22}\cdot 5^6\cdot 7^6(\doteqdot 7.710\times 10^{15})$
			$\pm 2M$	$2^{22}\cdot 5^6\cdot 7^6(\doteqdot 7.710\times 10^{15})$
$Q(\sqrt{21},\sqrt{6})$	168	$(3+\sqrt{6})(4+\sqrt{14})(7+\sqrt{21})$	$\pm M$	$2^{22}\cdot 3^6\cdot 7^6(\doteqdot 3.597\times 10^{14})$
			$\pm 2M$	$2^{22}\cdot 3^6\cdot 7^6(\doteqdot 3.597\times 10^{14})$
$Q(\sqrt{17},\sqrt{2})$	136	$(2+\sqrt{2})(17+3\sqrt{17})$	$\pm M$	$2^{22}\cdot 17^6(\doteqdot 1.012\times 10^{14})$
			$\pm 3M$	$2^{22}\cdot 17^6\cdot 3^4(\doteqdot 8.200\times 10^{15})$
$Q(\sqrt{33},\sqrt{2})$	264	$(2+\sqrt{2})(33+\sqrt{33})$	$\pm M$	$2^{22}\cdot 3^6\cdot 11^6(\doteqdot 5.417\times 10^{15})$

Table 3

K	$\sqrt{d_k}$	M	α	d_N	congruences mod 4
$Q(\sqrt{5},\sqrt{21})$	105	$\frac{5+\sqrt{5}}{2}\cdot\frac{21+\sqrt{21}}{2}$	$M,-3M$ $-M,\ 3M$ $\pm 2M,\pm 6M$	$5^6\cdot 3^6\cdot 7^6(\doteqdot 1.340\times 10^{12})$ $5^6\cdot 3^6\cdot 7^6\cdot 2^8(\doteqdot 3.340\times 10^{14})$ $5^6\cdot 3^6\cdot 7^6\cdot 2^{12}(\doteqdot 5.489\times 10^{15})$	$\frac{5+\sqrt{5}}{2}\equiv-(\frac{1-\sqrt{5}}{2})^2$ $\frac{21+\sqrt{21}}{2}\equiv-(\frac{1-\sqrt{21}}{2})^2$
$Q(\sqrt{5},\sqrt{29})$	145	$\frac{5+\sqrt{5}}{2}\cdot\frac{29+3\sqrt{29}}{2}$	M $-M$ $-3M$	$5^6\cdot 29^6(\doteqdot 9.294\times 10^{12})$ $5^6\cdot 29^6\cdot 2^8(\doteqdot 2.379\times 10^{15})$ $5^6\cdot 29^6\cdot 3^4(\doteqdot 7.528\times 10^{14})$	$\frac{5+\sqrt{5}}{2}\equiv-(\frac{1-\sqrt{5}}{2})^2$ $\frac{29+3\sqrt{29}}{2}\equiv-(\frac{1-3\sqrt{29}}{3})^2$
$Q(\sqrt{5},\sqrt{41})$	205	$\frac{15+\sqrt{205}}{2}\cdot\frac{41+5\sqrt{41}}{2}$	M $-3M$	$5^6\cdot 41^6(\doteqdot 7.422\times 10^{13})$ $5^6\cdot 41^6\cdot 3^4(\doteqdot 6.011\times 10^{15})$	$\frac{15+\sqrt{205}}{2}\equiv(\frac{1+\sqrt{205}}{2})^2$ $\frac{41+5\sqrt{41}}{2}\equiv(\frac{1-3\sqrt{41}}{2})^2$
$Q(\sqrt{5},\sqrt{61})$	305	$\frac{5+\sqrt{5}}{2}\cdot\frac{183+7\sqrt{61}}{2}$	$-M$	$5^6\cdot 61^6(\doteqdot 8.050\times 10^{14})$	$\frac{5+\sqrt{5}}{2}\equiv-(\frac{1-\sqrt{5}}{2})^2$ $\frac{183+7\sqrt{61}}{2}\equiv(\frac{1-\sqrt{61}}{2})^2$
$Q(\sqrt{5},\sqrt{69})$	345	$\frac{5+\sqrt{5}}{2}\cdot\frac{69+7\sqrt{69}}{2}$	$M,-3M$	$5^6\cdot 3^6\cdot 23^6(\doteqdot 1.686\times 10^{15})$	$\frac{5+\sqrt{5}}{2}\equiv-(\frac{1-\sqrt{5}}{2})^2$ $\frac{69+7\sqrt{69}}{2}\equiv-(\frac{1+\sqrt{69}}{2})^2$
$Q(\sqrt{5},\sqrt{89})$	445	$\frac{5+\sqrt{5}}{2}\cdot\frac{89+3\sqrt{89}}{2}$	$-M$	$5^6\cdot 89^6(\doteqdot 7.765\times 10^{15})$	$\frac{5+\sqrt{5}}{2}\equiv-(\frac{1-\sqrt{5}}{2})^2$ $\frac{89+3\sqrt{89}}{2}\equiv(\frac{1+3\sqrt{89}}{2})^2$
$Q(\sqrt{13},\sqrt{17})$	221	$\frac{13+3\sqrt{13}}{2}\cdot\frac{119+25\sqrt{17}}{2}$	M $-3M$	$13^6\cdot 17^6(\doteqdot 1.165\times 10^{14})$ $13^6\cdot 17^6\cdot 3^4(\doteqdot 9.437\times 10^{15})$	$\frac{13+3\sqrt{13}}{2}\equiv-(\frac{3-\sqrt{13}}{2})^2$ $\frac{119+25\sqrt{17}}{2}\equiv-(\frac{1-\sqrt{17}}{2})^2$
$Q(\sqrt{13},\sqrt{29})$	377	$\frac{143+23\sqrt{13}}{2}\cdot\frac{29+5\sqrt{29}}{2}$	$-M$	$13^6\cdot 29^6(\doteqdot 2.871\times 10^{15})$	$\frac{143+23\sqrt{13}}{2}\equiv-(\frac{1-\sqrt{13}}{2})^2$ $\frac{29+5\sqrt{29}}{2}\equiv(\frac{1+3\sqrt{29}}{2})^2$
$Q(\sqrt{17},\sqrt{21})$	357	$\frac{17+\sqrt{17}}{2}\cdot\frac{63+11\sqrt{21}}{2}$	$M,-3M$	$17^6\cdot 7^6\cdot 3^6(\doteqdot 2.070\times 10^{15})$	$\frac{17+\sqrt{17}}{2}\equiv(\frac{1+\sqrt{17}}{2})^2$ $\frac{63+11\sqrt{21}}{2}\equiv(\frac{3+\sqrt{21}}{2})^2$
$Q(\sqrt{21},\sqrt{33})$	231	$\frac{3+\sqrt{21}}{2}\cdot\frac{-11+3\sqrt{33}}{2}$ $\frac{7+\sqrt{77}}{2}$	$M,-3M$	$3^6\cdot 7^6\cdot 11^6(\doteqdot 1.519\times 10^{14})$	$\frac{3+\sqrt{21}}{2}\equiv(\frac{1+\sqrt{21}}{2})^2$ $\frac{-11+3\sqrt{33}}{2}\equiv(\frac{3+\sqrt{33}}{2})^2$ $\frac{7+\sqrt{77}}{2}\equiv(\frac{1+\sqrt{77}}{2})^2$

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