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PETTIS MEAN CONVERGENCE OF MARTINGALES

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0. Introduction

Convergence of Bochner integrable martingales has been studied by many authors [1,4,6,10]. For a Banach space with Radon–Nikodym property, the basic theory of convergence of Bochner integrable martingales is very similar to the theory of convergence of scalar valued martingales. J.J.Uhl, Jr. [9] studied the Pettis mean convergence of Pettis integrable martingales and the stated that in the case of Pettis integrable martingales there seems to be no nontrivial conditions one can place on Banach space to obtain a simple theory of convergence (see [9] p.374). As a related problem, in this paper, we characterize the Banach space with Radon–Nikodym property in terms of Pettis mean convergence of Pettis integrable martingales.

The first section is concerned with preliminaries which establish the setting for the work which follows. The second section is devoted to the characterization of Banach space with Radon–Nikodym property in terms of Pettis mean convergence of martingale.

1. Preliminaries

Throughout this paper (Ω, Σ, μ) is a fixed probability space. E is a Banach space with continuous dual E^*. A function $X : \Omega \to E$ is strongly measurable if X is the almost everywhere $[\mu]$ limit of simple measurable functions of the form $\sum_{i=1}^{n} x_i^1 A_i$, $x_i \in E$, $A_i \in \Sigma$. A random variable means a strongly measurable function. A random variable $X : \Omega \to E$ is called Pettis integrable if $f(X) \in L_1(\Omega, \Sigma, \mu)$ for all $f \in E^*$

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and for each $A \in \Sigma$ there exists $x_A \in E$ satisfying the identity $f(x_A) = \int_A f(X)d\mu$ for all $f \in E^*$. In this case one writes $x_A = \int_A X d\mu$. If also $\|X\| \in L_1(\mu)$ then f is Bochner integrable. If random variable X is Pettis integrable, the Pettis norm of X is

$$\|X\| = \sup_{\|f\| \leq 1} \int_\Omega |f(X)| d\mu.$$

It is known that $\|X\|$ is equal to the semivariation of the measure $A \mapsto \int_A X d\mu$, i.e.,

$$\|X\| = \sup \| \sum \alpha_i \int_{A_i} X d\mu \|$$

where the supremum is taken over all finite collections of scalars with $|\alpha_i| \leq 1$ and all partitions of Ω into finitely many disjoint sets. One also has

$$\|X\| \leq 4 \sup_{A \in \Sigma} \| \int_A X d\mu \|$$

this is known result, which can be proved as follows : Let $X \cdot \mu$ be the measure defined by

$$(X \cdot \mu)(A) = \int_A X d\mu.$$

Then

$$\begin{aligned} \|X\| &= \sup_{\|f\| \leq 1} \{\text{Variation of } f(X) \cdot \mu\} \\ &\leq \sup_{\|f\| \leq 1} 4 \sup_A | \int_A f(X) d\mu | \\ &= 4 \sup_A | \sup_{\|f\| \leq 1} |f(\int_A X d\mu)|| \\ &= 4 \sup_A \| \int_A X d\mu \| \end{aligned}$$

Thus the norm of X given by $\sup_A \| \int_A X d\mu \|$ is equivalent to Pettis norm. After identification of functions which agree on all but possibly a of μ–measure zero, the collection of all E–valued strongly measurable and Pettis integrable functions becomes a normed linear space which is typically incomplate [8]. This space will be denoted by $P_1(E)$ or $P_1(\mu, E)$ or $P_1(\Sigma, \mu, E)$ depending on the context. The Banach space consisting of E–valued strongly measurable and Bochner integrable functions will be denoted by $L_1(E)$ or $L_1(\mu, E)$ or $L_1(\Sigma, \mu, E)$.

DEFINITION 1.1 [9]. *Let $X \in P_1(\Sigma, E)$ and Σ_0 be a sub–σ–field of Σ. $Y \in P_1(\Sigma_0, E)$ is called the conditional expectation of X with respect to Σ_0 if $\int_E X d\mu = \int_E Y d\mu$ for all $E \in \Sigma_0$. In this case one writes $E(X/\Sigma_0) = Y$.*

DEFINITION 1.2. *Let J be a directed set filtering to the right and $\{\Sigma_t, t \in J\}$ be an increasing net of sub–σ–algebras of Σ; i.e., $t_1 \leq t_2$ implies $\Sigma_{t_1} \subseteq \Sigma_{t_2}$. $(X_t, \Sigma_t, t \in J) \subset P_1(\Sigma, E)$ is a martingale if $t_1 \leq t_2$ implies $E(X_{t_2}|\Sigma_{t_1}) = X_{t_1}$ and $X_t \in P_1(\Sigma_t, E)$.*

A function $F : \Sigma \to E$ is called a vector measure if $F(\cup A_n) = \Sigma F(A_n)$ for every sequence (A_n) of disjoint sets in Σ. The measure F is μ–continuous iff $F(A) = 0$ for all $A \in \Sigma$ with $\mu(A) = 0$. The variation of vector measure F is the extended non–negative function $|F|$ whose value on a set $E \in \Sigma$ is given by

$$|F|(E) = \sup_{\pi} \sum_{A \in \pi} \|F(A)\|$$

where the supremum is taken over all partitions π of E into finite disjoint sequence $A_1, \cdots, A_n$ of Σ. The measure F has a finite variation if $|F|(\Omega) < \infty$. The measure F has a σ–finite variation if Ω is a countable union of sets on which F has finite variation.

DEFINITION 1.3 [4]. *The Banach space E is said to have the Radon–Nikodym property iff for every probability space (Ω, Σ, μ) and every measure $F : \Sigma \to E$ such that F is μ–continuous and F has finite variation on Ω, there is a Bochner integrable $X : \Omega \to E$ such that $F(A) = \int_A X d\mu$ for all $A \in \Sigma$. It follows that if F is merely required to have σ–finite variation on Ω, then there is a Pettis integrable strongly measurable function $X : \Omega \to E$ such that $F(A) = \int_A X d\mu$ for all*

$A \in \Sigma$. Notice that for a Pettis integrable strongly measurable function $X : \Omega \to E$, if measure F given by $F(A) = \int_A X d\mu$ has a finite variation then X is a Bochner integrable.

2. Pettis mean convergence of martingales and Radon–Nikodym property

For a E–valued stochastic process $(X_t, \Sigma_t, J) \subseteq P_1(\mu, E)$, we define vector–valued measure $X_t \cdot \mu : \Sigma \to E$ as follows ;

$$(X_t \cdot \mu)(A) = \int_A X_t d\mu$$

If X_t is Bochner integrable then $X_t \cdot \mu$ has a finite variation and $|X_t \cdot \mu|(\Omega) = \int_\Omega \|X_t\| d\mu = \|X_t\|$ 1. Thus we have that $\sup_t |X_t \cdot \mu|(\Omega) = \sup_t \|X_t\|$ 1. In other word, the finiteness of set function $\sup_t |X_t \cdot \mu|$ is the same as L_1–boundedness of (X_t).

J:J. Uhl [12] showed that Banach space E has Radon–Nikodym property iff for every probability space (Ω, Σ, μ), uniformly integrable martingale $(X_t, \Sigma_t, J) \subseteq L_1(\mu, E)$ with finiteness of set function $\sup_t |X_t \cdot \mu|$ converges in $L_1(\mu, E)$–norm. Here uniformly integrability of $(X_t) \subseteq L_1(\mu, E)$ means $\lim_{\mu(A)\to 0} \int_A \|X_t\| d\mu$ uniformly in $t \in J$.

THEOREM 2.1. *Banach space E has Radon–Nikodym property iff for every probability space (Ω, Σ, μ), uniformly integrable martingale $(X_t, \Sigma_t, J) \subseteq L_1(\mu, E)$ with σ–finiteness of set function $\sup_t |X_t \cdot \mu|$ on (Ω, Σ_{t_0}) for some $t_0 \in J$ converges in $L_1(\mu, E)$–norm.*

Proof. Since $\sup_t |X_t \cdot \mu|$ is σ–finite on (Ω, Σ_{t_0}), there exists a sequence $\{A_n\} \subseteq \Sigma_{t_0}$ of disjoint sets such that $\sup_t |X_t \cdot \mu|(A_n) < +\infty$. Uniformly integrability assumption implies that there exists n_0 such that

$$\int_{(\bigcup_{n \geq n_0} A_n)^c} \|X_t\| d\mu < \varepsilon \quad \text{for all} \quad t \in J.$$

Let

$$A = \bigcup_{n=1}^{n_0} A_n.$$

Then A belongs to Σ_{t_0} amd $(X_t 1_A,\ \Sigma_t \cap A,\ t \geq t_0)$ is uniformly integrable martingale on $(A,\ \Sigma \cap A,\ \mu|A)$ and $\sup |X_t \cdot \mu|(A) < +\infty$. By the above Uhl's theorem, $(X_t 1_A)$ is Cauchy in $L_1(\mu, E)$. For $t_1, t_2 \geq t_0$, we have

$$\begin{aligned}\int_\Omega \|X_{t_1} - X_{t_2}\| d\mu &= \int_A \|X_{t_1} - X_{t_2}\| d\mu + \int_A c\|X_{t_1} - X_{t_2}\| d\mu \\ &\leq \int_A \|X_{t_1} - X_{t_2}\| d\mu + 2\varepsilon.\end{aligned}$$

The completeness of $L_1(\mu, E)$ implies that (X_t) converges in $L_1(\mu, E)$–norm.

J.J. Uhl [9] gave an example in which $P_1(\mu, E)$–bounded uniformly integrable martingale (X_t) of Pettis integrable functions (valued in a reflexive Banach space) does not converge in $P_1(\mu, E)$–norm. To obtain the Pettis mean convergence theorem of Pettis integrable martingale, we need a stronger condition than $P_1(\mu, E)$–boundedness. The proof of the next theorem is a modification of that of Uhl's Theorem [12]. Uniformly integrability of $(X_t) \subseteq P_1(\mu, E)$ means $\lim_{\mu(E) \to 0} \int_E X_t d\mu = 0$ uniformly in $t \in J$, i.e., given $\varepsilon > 0$ there exist $\delta > 0$ such that $\mu(A) < \delta$ implies

$$\| \int_A X_t d\mu \| < \varepsilon \quad \text{for all} \quad t \in J. \quad \text{(see [11])}$$

This definition of uniformly integrability is reduced to the ordinary definition of uniformly integrability in the case of real–valued functions.

THEOREM 2.2. *The followings are equivalent.*

(i) *Banach space E has Rodon–Nikodym property.*
(ii) *For every probability space (Ω, Σ, μ), every uniformly integrable martingale $(X_t, \Sigma_t, J) \subseteq P_1(\mu, E)$ converges in $P_1(\mu, E)$–norm if set function $\sup_t |X_t \cdot \mu|$ is σ–finite on (Ω, Σ_{t_0}) for some $t_0 \in J$.*

Proof. (i) $\Longrightarrow$ (ii). For $A \in \cup_t \Sigma_t$ set $F(A) = \lim_t \int_A X_t d\mu$. Since (X_t, Σ_t, J) is uniformly integrable, it follows that $\lim_{\mu(E)\to 0} F(E) = 0$ on $\cup_{t\in J}\Sigma_t$. On the other hand, since set function $\sup_t |X_t \cdot \mu|$ is σ–finite on (Ω, Σ_{t_0}) for some $t_0 \in J$, there exists a partition $\{B_n\}$ of Ω into disjoint sets in Σ_{t_0} such that $\sup |X_t \cdot \mu|(B_n) < \infty$ for all n. For a fixed n, if $\pi \subseteq \cup_t \Sigma_t$ is a finite partition of B_n then there is an index $t_1 > t_0$ such that $\pi \subseteq \Sigma_{t_1}$. Consequently one has

$$\begin{aligned}\sum_{A\in\pi} \|F(A)\| &= \sum_{A\in\pi} \| \int_A X_{t_1} d\mu \| \\ &= \sum_{A\in\pi} \|(X_{t_1} \cdot \mu)(A)\| \leq |X_{t_1} \cdot \mu|(B_n) \\ &\leq \sup_t |X_t \cdot \mu|(B_n) < +\infty.\end{aligned}$$

Hence F has σ–finite variation on $\cup_t \Sigma_t$. An appeal to [2] produces a μ–continuous vector measure G of σ–finite variation on Σ_0, the σ–field generated by $\cup_t \Sigma_t$, such taht $G(E) = F(E)$ on $\cup_t \Sigma_t$. Since E has the Radon–Nikodym property, there is $X \in P_1(\mu|\Sigma_0, E)$ such that $G(A) = \int_A X d\mu$ for all $A \in \Sigma_0$. But if $A \in \cup_t \Sigma_t$, then

$$\lim_t \int_A X_t d\mu = F(A) = G(A) = \int_A X d\mu.$$

By [9, Lemma 1.4] (X_t) converges to X in $P_1(\mu, E)$–norm.

(ii) $\Longrightarrow$ (i). Let (Ω, Σ, μ) be a fixed probability space and $F : \Sigma \to E$ be a μ–continuous vector measure of σ–finite variation. There exists a countable partition $\pi_0 = \{A_n\}$ of Ω such that $|F|(A_n) < +\infty$. Let J be the class of all partitions π of Ω into Σ which are refinements of π_0 then J is a directed set by refinement. For each $\pi \in J$, define

$$X_\pi = \sum_{A\in\pi} \frac{F(A)}{\mu(A)} 1_A \quad (\text{convention } \frac{0}{0} = 0)$$

Let Σ_π be the σ–field generated by π. By the countable additivity of F, we see that X_π is a Pettis integrable and $\int_\Omega X_\pi d\mu = \sum_{A\in\pi} F(A)$. Simple calculation shows that (X_π, Σ_π, J) is a martingale in $P_1(\mu, E)$. Next we show that $F(A) = \lim_\pi \int_A X_\pi d\mu$ for every $A \in \Sigma$. For $A \in \Sigma$, set $\pi' = \{A \cap A_i,\ A_i \backslash (A \cap A_i) | i = 1, 2, \cdots\}$. Then π' is a partition of Ω and is a refinement of π_0 and

$$\begin{aligned}\int_A X_{\pi'} d\mu &= \sum_{i=1}^{\infty} \int_{A \cap A_i} X_{\pi'} d\mu \\ &= \sum_{i=1}^{\infty} F(A \cap A_i) \\ &= F(A).\end{aligned}$$

Thus we obtain that $F(A) = \lim_\pi \int_A X_\pi d\mu$ for $A \in \Sigma$. Next we will show that $\sup_\pi |X_\pi \cdot \mu|(A_n) < +\infty$, for all n, so $\sup_\pi |X_\pi \cdot \mu|$ is σ–finite on (Ω, Σ_{π_0}). For a fixed n, let $B_1, \cdots, B_k$ be a partition of A_n and $B_i \in \Sigma_\pi$. Then we have

$$\begin{aligned}\sum_{i=1}^{k} \|X_\pi \cdot \mu)(B_i)\| &= \sum_{i=1}^{k} \| \int_{B_i} X d\mu \| \\ &= \sum_{i=1}^{k} \|F(B_i)\| \leq |F|(A_n)\end{aligned}$$

so

$$|X_\pi \cdot \mu|(A_n) \leq |F|(A_n) \quad \text{for all} \quad \pi \in J.$$

Thus we have $\sup_\pi |X_\pi \cdot \mu|(A_n) \leq |F|(A_n) < +\infty$. Also since $F \ll \mu$, we have $|F| \ll \mu$. Hence for each $\varepsilon > 0$ there is $\delta > 0$ such that $|F|(A) < \varepsilon$ whenever $\mu(A) < \delta$. Now if $A \in \Sigma_\pi$ and $\mu(A) < \delta$ then

$$\| \int_A X_\pi d\mu \| \leq |X_\pi \cdot \mu|(A) \leq |F|(A) < \varepsilon.$$

Thus (X_π, Σ_π, J) is uniformly integrable. By hypothesis (X_π) converges to $X \in P_1(\mu, E)$ in Pettis norm. Thus we have $F(A) = \lim_\pi \int_A X_\pi d\mu = \int_A X d\mu$. This completes the proof.

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