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TRANSFORMATION GROUPS WITH THE DENSE ACTION PROPERTY

CHINKU CHU

1. Introduction

In this paper, the dense action property is introduced in transformation groups. This is a special property of the topological phase group on the phase space in a given transformation group. We describe some consequences of the dense action property and some relationships between transformation groups with the dense action property and homomorphisms.

A *transformation group* is a pair (X,T) of a compact Hausdorff space X and a topological group T which acts on X. By this, we mean that there exists a continuous map $\pi : X \times T \to X$, $(x,t) \to xt$ such that $x(ts) = (xt)s$ for every $x \in X$ and $s,t \in T$, and also $xe = x$ for every $x \in X$, where e is the identity element of T. A closed nonempty subset M of X is said to be a *minimal set* if, for every $x \in M$, the orbit xT is a dense subset of M. If (Y,T) is also a transformation group, a *homomorphism* from (X,T) to (Y,T) is defined to be a continuous map $f : X \to Y$ such that $f(xt) = f(x)t$ for all $x \in X$ and $t \in T$.

Unless otherwise stated in this paper, all maps of transformation groups mean homomorphisms of transformation groups, all spaces are assumed to be Hausdorff, and most of the terms and notations come from [3].

2. The dense action property

DEFINITION 1. *A transformation group (X,T) has the dense action property if for every point x in X and every open set 0 in T, $\mathrm{cl}(x0)$ is a neighborhood of xt for all t in 0, i.e., $\mathrm{cl}(x0)$ is a neighborhood of $x0$ [2].*

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LEMMA 1. *Let (X,T) be a transformation group with the dense action property and $f:(X,T)\to(Y,T)$ be an open epimorphism of transformation groups. Then (Y,T) has the dense action property.*

Proof. Let 0 be an open subset of T and let y be in Y. Since f is an epimorphism there exists a point x in X with $f(x)=y$. The set $\mathrm{cl}(x0)$ is a neighborhood of xt for every t in 0 by the dense action property of (X,T). Also

$$f(\mathrm{cl}(x0))\subset \mathrm{cl}(f(x0))=\mathrm{cl}(f(x)0)=\mathrm{cl}(y0).$$

Since f is open, $f(\mathrm{cl}(x0))$ is a neighborhood of yt for all t in 0. Hence $\mathrm{cl}(y0)$ is a neighborhood of yt for all $t\in 0$. Therefore the transformation group (Y,T) has the dense action property.

PROPOSITION 1. *Let (X,T) and (Y,T) be transformation groups and a homomorphism $f:(X,T)\to(Y,T)$ be an onto local homeomorphism. Then (X,T) has the dense action property if and only if (Y,T) has the dense action property.*

Proof. Assume that (Y,T) has the dense action property. Let x be a point in X with $y=f(x)$. Since $f:X\to Y$ is an onto local homeomorphism there exist open neighborhoods U of x in X and V of y in Y such that $f|_U:U\to V$ is a homeomorphism. Since X is a compact Hausdorff, it is regular. Hence there exists an open neighborhood U_1 of x such that $\mathrm{cl}\,U_1\subset U$. Thus $f|_{\mathrm{cl}\,U_1}:\mathrm{cl}\,U_1\to f(\mathrm{cl}\,U_1)$ is a homeomorphism, $f(\mathrm{cl}\,U_1)$ is a neighborhood of y, and there exists an open neighborhood G of e in T such that $\mathrm{cl}(yG)$ is contained in $f(\mathrm{cl}\,U_1)$.

Let

$$U_x=(f|_{\mathrm{cl}\,U_1})^{-1}(\mathrm{Int}(\mathrm{cl}(yG))).$$

Then U_x is an open neighborhood of x, since y is in $\mathrm{Int}(\mathrm{cl}(yG))$. Since $\mathrm{cl}(yG)$ is contained in $f(\mathrm{cl}\,U_1)$, U_x is contained in $\mathrm{cl}\,U_1$. Hence $\mathrm{cl}\,U_x$ is contained in $\mathrm{cl}\,U_1$. Thus we have the homeomorphism $f|_{\mathrm{cl}\,U_x}:\mathrm{cl}\,U_x\to \mathrm{cl}(f(U_x))$. It is clear that

$$\mathrm{cl}(f(U_x))=\mathrm{cl}(\mathrm{Int}(\mathrm{cl}(yG)))=\mathrm{cl}(yG),$$

since $\mathrm{Int}(\mathrm{cl}(yG))$ contains yG. Thus we have the homeomorphism $f|_{\mathrm{cl}\,U_x}:\mathrm{cl}\,U_x\to\mathrm{cl}(yG)$. Since the map $\Pi_x:T\to X$ defined by $\Pi_x(t)=xt$

for all $t \in T$ is continuous, there exists an open neighborhood G_1 of e in T with $xG_1 \subset \mathrm{cl}\, U_x$. Then

$$f(\mathrm{cl}(xG_1)) = \mathrm{cl}(f(xG_1)) = \mathrm{cl}(f(x)G_1).$$

So $f(\mathrm{cl}(xG_1)) = \mathrm{cl}(f(x)G_1)$, which is a neighborhood of $f(x) = y$, since the open set G_1 contains e of T. Since $\mathrm{cl}(xG_1)$ is homeomorphic to $\mathrm{cl}(f(x)G_1)$, the set $\mathrm{cl}(xG_1)$ is a neighborhood of x.

Let $x \in X$ and 0 be an open set in T. We will show that $\mathrm{cl}(x0)$ is a neighborhood of $x0$. For any point $z \in x0$, let $z = xt_0$ for some t_0 in 0. There exists an open neighborhood W of e such that t_0W is contained in 0 and $\mathrm{cl}(xt_0W)$ is contained in $\mathrm{cl}(x0)$. We may assume that $f|_{\mathrm{cl}(xt_0W)} : \mathrm{cl}(xt_0W) \to \mathrm{cl}(yt_0W)$ is a homeomorphism, where $y = f(x)$. Since $\mathrm{cl}(yt_0W) = \mathrm{cl}(f(z)W)$ is a neighborhood of $f(z) = f(x)t_0$, $\mathrm{cl}(zW)$ is a neighborhood of z. This tells us that $\mathrm{cl}(x0)$ is a neighborhood of z for all z in $x0$.

Conversely, assume that (X,T) has the dense action property. Then by the Lemma 1, (Y,T) has the dense action property.

COROLLARY 1. *Let $f : (X,T) \to (Y,T)$ be a covering map of transformation groups. Then (X,T) has the dense action property if and only if (Y,T) has the dense action property.*

PROPOSITION 2. *Let (X,T) be a transformation group. Let (Y,T) be a transformation group with the dense action property. Then every epimorphism of (X,T) onto (Y,T) is open.*

Proof. Let $f : (X,T) \to (Y,T)$ be an epimorphism of transformation groups. Let U be an open subset of X. Now for every $x \in U$,

$$x\Pi_x^{-1}(U) = \Pi_x(\Pi_x^{-1}(U)) \subset U,$$

where $\Pi_x : T \to X$ is defined by $\Pi_x(t) = xt$ for all $t \in T$. Since $\Pi_x^{-1}(U)$ contains the identity of T for every x in U, x is in $x\Pi_x^{-1}(U)$. Hence U is contained in $\cup\{x\Pi_x^{-1}(U) \mid x \in U\}$. Clearly, $\cup\{x\Pi_x^{-1}(U) \mid x \in U\}$ is contained in U. Thus we have

$$U = \cup\{x\Pi_x^{-1}(U) \mid x \in U\}.$$

Then

$$\begin{aligned}\mathrm{cl}(f(U)) &= \mathrm{cl}\left(f(\cup\{x\Pi_x^{-1}(U)\,|\,x\in U\})\right)\\ &= \mathrm{cl}\left(\cup\{f(x\Pi_x^{-1}(U))\,|\,x\in U\}\right)\\ &= \mathrm{cl}\left(\cup\{f(x)\Pi_x^{-1}(U)\,|\,x\in U\}\right),\end{aligned}$$

and

$$\cup\{\mathrm{cl}(f(x)\Pi_x^{-1}(U)|x\in U\}\subset \mathrm{cl}(\cup\{f(x)\Pi_x^{-1}(U)|x\in U\}).$$

The map Π_x is clearly continuous, so $\Pi_x^{-1}(U)$ is open in T. Also $\mathrm{cl}(f(x)\Pi_x^{-1}(U))$ is a neighborhood of $f(x)e = f(x)$, since (Y,T) has the dense action property and $\Pi_x^{-1}(U)$ contains e for every x in U. Hence $\cup\{\mathrm{cl}(f(x)\Pi_x^{-1}(U)|x\in U\}$ is a neighborhood of $f(x)$ for all x in U. Since $\cup\{\mathrm{cl}(f(x)\Pi_x^{-1}(U))|x\in U\}$ is contained in $\mathrm{cl}(f(U))$, $\mathrm{cl}(f(U))$ is a neighborhood of $f(x)$ for all x in U which means that $\mathrm{cl}(f(U))$ is a neighborhood of $f(U)$.

Now consider a point $f(x)$ in $f(U)$ for x in U. Since X is a compact Hausdorff space, it is regular, and there exists an open set V in X such that $x\in V\subset \mathrm{cl}\,V\subset U$. It is clear that $f(\mathrm{cl}\,V) = \mathrm{cl}(f(V))$. Now $\mathrm{cl}(f(V))$ is a neighborhood of $f(x)$ and $f(\mathrm{cl}\,V)\subset f(U)$. Thus $f(U)$ is a neighborhood of $f(x)$ for every x in U. Therefore $f(U)$ is open in Y. Hence we have proved the proposition.

COROLLARY 2. *Let $(X,\mathbf{Q})$ be a transformation group, where $\mathbf{Q}$ is the additive topological group of all rational numbers. Let $(S^1,\mathbf{Q})$ be the transformation group with the action $(e^{i\theta}, t)\to e^{i(\theta+t)}$, where S^1 be the unit circle. Then every homomorphism $f:(X,\mathbf{Q})\to(S^1,\mathbf{Q})$ is open.*

Proof. The transformation group $(S^1,\mathbf{Q})$ has the dense action property. By Proposition 2, f is open.

LEMMA 2. *Let (X,T) be a transformation group with the dense action property. If S is a dense subgroup of T, then (X,S) is a transformation group with the dense action property.*

Proof. Let G be an open set in S. Then there exists an open subset G' in T such that $G = G'\cap S$. For any point x in X, $x(\mathrm{cl}\,G)$ is contained in $\mathrm{cl}(xG)$. So $\mathrm{cl}(x(\mathrm{cl}\,G))$ is contained in $\mathrm{cl}(xG)$. Let $t'\in G'$. Since S

is dense in T, G' contains an element t of S. Hence t is in G, since $G = G' \cap S$. This means that every neighborhood of t' in T contains an element of G. So $t' \in \operatorname{cl} G$, i.e., G' is contained in $\operatorname{cl} G$. Hence $\operatorname{cl}(xG')$ is contained in $\operatorname{cl}(x(\operatorname{cl} G))$. So $\operatorname{cl}(xG')$ is contained in $\operatorname{cl}(xG)$. Thus we have $\operatorname{cl}(xG) = \operatorname{cl}(xG')$. By the dense action property of (X,T), $\operatorname{cl}(xG')$ is a neighborhood of xG. Hence $\operatorname{cl}(xG)$ is a neighborhood of xG.

PROPOSITION 3. *Let (X,T) be a transformation group with the dense action property. If S is a subgroup of T with finite index in T, then the transformation group (X,S) has the dense action property.*

Proof. From the fact that S is a subgroup of T and S has finite index in T, it follows that $\operatorname{cl} S$ is a subgroup with finite index in T. Since T is an union of finite cosets of $\operatorname{cl} S$ in T and each coset of $\operatorname{cl} S$ is closed, $\operatorname{cl} S$ is open in T. Hence the transformation group $(X, \operatorname{cl} S)$ has the dense action property. By Lemma 2. (X,S) is a transformation group with the dense action property.

Let (X,T) be a transformation group. We define $\pi^t(x) = xt$ for $t \in T$ and $x \in X$. Then for each $t \in T$, Π^t is a map of X into X, hence an element of the compact Hausdorff space X^X. The *enveloping semigroup* $E(X)$ or $E(X,T)$ of (X,T) is the closure of $\{\Pi^t \mid t \in T\}$ in X^X. Let $q \in E(X)$. Define $L_q : (E(X),T) \to (E(X),T)$ by $L_q(p) = qp$ for all $p \in E(X)$. Then L_q is an endomorphism of the transformation group $(E(X),T)$. The two points x and y in X are *distal* if either $x = y$ or if there is not any net (t_i) in T such that $\lim xt_i = \lim zt_i$. A transformation group (X,T) is said to be *distal* if every two points of X are distal. A homomorphism $f : (X,T) \to (Y,T)$ of transformation groups is called *distal* if every two points in $f^{-1}(y)$ are distal for all y in Y.

LEMMA 3. *Let $f : (X,T) \to (Y,T)$ be a distal homomorphism of minimal transformation groups. Then f is an open homomorphism [1].*

The following theorem gives a sufficient condition for a transformation group to have the dense action property.

PROPOSITION 4. *Let (X,T) be a distal transformation group such that the map $f : T \to \{\Pi^t | t \in T\}$ defined by $f(t) = \Pi^t$, for all $t \in T$, is* *open, where $\{\Pi^t | t \in T\}$ is a subspace of X^X. Then every minimal set of (X,T) has the dense action property.*

Proof. Let 0 be an open subset of T. Since f is open, $f(0)$ is open in $f(T)$. Hence there exists an open set U in $E(X)$ such that $f(0) = f(T) \cap U$. It is clear that $\mathrm{cl}(f(0))$ is contained in $\mathrm{cl}\, U$. Let $p \in \mathrm{cl}\, U$. For any open neighborhood V of p in $E(X)$, V contains a point q of U. So $U \cap V$ is a nonempty open set in $E(X)$. Hence $U \cap V$ contains a point r of $f(T)$, and $r \in U \cap f(T) = f(0)$. Thus the open neighborhood V of p contains a point r of $f(0)$. So $p \in \mathrm{cl}(f(0))$, i.e., $\mathrm{cl}\, U$ is contained in $\mathrm{cl}(f(0))$. Therefore $\mathrm{cl}(f(0)) = \mathrm{cl}\, U$. Since U is an open neighborhood of $f(0)$, $\mathrm{cl}(f(0))$ is a neighborhood of $f(0)$ in $E(X)$. Now, given an open set $0'$ in T and a point p' in $E(X)$,

$$\mathrm{cl}(p'f(0') = \mathrm{cl}(L_{p'}(f(0'))) = L_{p'}(\mathrm{cl}(f(0'))),$$

since $L_{p'} : (E(X),T) \to (E(X),T)$ is a homomorphism of the compact space $E(X)$. Since (X,T) is distal, $E(X)$ is a group. Hence $L_{p'}$ is an isomorphism of $(E(X),T)$. Hence $L_{p'}(\mathrm{cl}(f(0')))$ is a neighborhood of $L_{p'}(f(0')) = p'f(0')$. This means that $(E(X),T)$ is a transformation group with the dense action property.

Let (M,T) be a minimal transformation group of (X,T) and $x \in M$. Define $\theta_x : (E(X),T) \to (X,T)$ by $\theta_x(p) = xp$ for all $p \in E(X)$. Then θ_x is a homomorphism. Since $E(X)$ is a minimal set, $\theta_x(E(X))$ is also minimal in (X,T). Clearly, $x \in \theta_x(E(X))$. So $\theta_x(E(X)) = M$. Since (X,T) is distal, $(E(X),T)$ is distal. Hence $\theta_x : (E(X),T) \to (M,T)$ is distal, and thus is open by Lemma 3. Since $(E(X),T)$ has the dense action property, Lemma 1 implies that (M,T) has the dense action property. This completes the proof.

EXAMPLE. *Let S^n be the unit n–sphere, $n \geq 1$, and $GL(n+1,\mathbf{R})$ be the group of all nonsingular $(n+1) \times (n+1)$ matrices with coefficients in $\mathbf{R}$. Define an action $\Pi : S^n \times GL(n+1, \mathbf{R}) \to S^n$ by*

$$\Pi(x,A) = \frac{xA}{|xA|}$$

for all $x \in S^n$ and $A \in GL(n+1), \mathbf{R})$. Then $(S^n, GL(n+1, \mathbf{R}))$ is a minimal transformation group with the dense action property. Let

$GL(n+1,\ \mathbf{Q})$ be the group of all non–singular $(n+1) \times (n+1)$ matrices whose entries are rational numbers. Since $GL(n+1, \mathbf{Q})$ is a dense subgroup of $GL(n+1, \mathbf{R})$, the transformation group $(S^n,\ GL(n+1, \mathbf{Q})$ has the dense action property by Lemma 2. Let $SL(n+1,\ \mathbf{Q}) = \{A | A \in GL(n+1,\ \mathbf{Q}),\ det\ A > 0\}$. Then $SL(n+1, \mathbf{Q})$ is a subgroup of $GL(n+1, \mathbf{Q})$ and the index is $[GL(n+1, \mathbf{Q}) : SL(n+1, \mathbf{Q})] = 2$. So from Proposition 3, the transformation group $(S^n,\ SL(n+1, \mathbf{Q}))$ has the dense action property.

References

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Chungnam University
Daejeon 302–764, Korea