Comm. Korean Math. Soc. 4(1989), No. 1, pp. 129~138

A GENERALIZATION OF CONTINUOUS POSETS *

Y.H. HAN, S.S. HONG, C.K. LEE AND P.U. PARK

0. Introduction

Since the concept of continuous lattices has been introduced by Scott [12], it has played an important role in the study of various mathematical structures, e.g., logic, computer sciences, topological structures, topological semilattices and algebraic structures (see [1], [2], [3], [8], [10], [12]).

Instead of complete lattices, Hoffmann generalized this concept to that of continuous posets and obtained many interesting results ([3], [6], [7], [8], [10]).

Recently, Lee [9] further introduced a concept of countably approximating lattices which properly contains that of continuous lattices and showed that this new larger class enjoyed almost all properties of continuous lattices.

The purpose of this paper is to introduce another class of posets which generalizes countably approximating lattices and continuous posets and to extend those properties of above two classes to the new class.

Using the countably way below relation [9] and σ-ideals, we introduce countably approximating posets as those posets A with countably directed joins such that the join map $\vee : \sigma Idl(A) \to A$ has a left adjoint.

We then show that countably approximating posets are precisely retracts of σ-algebraic posets by maps preserving countably directed joins and that the countably way below relation on a countably approximating poset satisfies the interpolation property.

Furthermore, introducing σ-Scott topology on a poset, we characterize countably approximating posets by those posets whose σ-Scott open sets can be determined by countably way below relations.

Received December 27, 1988.

*This research was supported by a grant from the Korea Science and Engineering Foundation.

For the terminology not introduced in this paper, we refer to [3], [9], [10].

1. Countably approximating posets

In this section, we introduce countably approximating posets and characterize them by σ–ideals.

We recall that a subset I of a poset A is said to be a σ–ideal if I is a lower set and countably directed.

For a poset A, the set of σ–ideals of A will be denoted by $\sigma Idl(A)$ and then $\sigma Idl(A)$ with the inclusion relation is obviously a poset which has countably directed joins. Indeed, for a countably directed sequence (I_n) in $\sigma Idl(A)$, the join is simply $\cup I_n$.

If A has countable joins, then $\sigma Idl(A)$ is closed under arbitrary intersections; hence $\sigma Idl(A)$ is a complete lattice.

Every poset A can be embedded in $\sigma Idl(A)$ by the principal ideal map $A \to \sigma Idl(A)$ $(a \mapsto \downarrow a)$. Furthermore, one can easily show that a poset A has countably directed joins iff the map $\downarrow\_ : A \to \sigma Idl(A)$ has a left adjoint.

In the sequel, we will assume that *every poset has countably directed joins.*

DEFINITION 1.1. *For elements a, b of a poset A, we say that a is countably way below b, in symbol $a \ll_c b$ provided for any σ–ideal I with $a \leq \vee I$, one has $b \in I$.*

The following is immediate from the fact that for any countably directed subset S of a poset A, $\downarrow S$ is a σ–ideal in A.

REMARK. *Let a, b be elements of a poset A, then $a \ll_c b$ iff for any countably directed subset S of A with $a \leq \vee S$, there is an $s \in S$ with $b \leq s$.*

Furthermore, if A has countable joins, then $a \ll_c b$ iff for any subset S of A with $a \leq \vee S$, there is a countable subset C of S with $b \leq \vee C$.

DEFINITION 1.2. *An element a of a poset A is said to be a Lindelöf element if whenever $a \leq \vee B$ $(B \subseteq A)$, there is a countable subset C of B with $a \leq \vee C$.*

We note that for the open set lattice $0(X)$ of a topological space X, $A \ll_c B$ in $0(X)$ if there is a Lindelöf subspace L of X with $A \subseteq L \subseteq B$. Furthermore, an element A of $0(X)$ is a Lindelöf element iff A is a Lindelöf subspace.

The following is immediate and we omit the proof (also see [9]).

PROPOSITION 1.3. *Let A be a poset, then one has :*

1) *If $b \ll_c a$ in A, then $b \leq a$.*
2) *If $b' \leq b \ll_c a \leq a'$ in A, then $b' \ll_c a'$.*
3) *If a is a Lindelöf element of A, then $a \ll_c a$. Moreover, if A has countable joins and $a \ll_c a$, then a is a Lindelöf element of A.*
4) *If A has countable joins, then for any $a \in A$, $\downarrow_c a = \{x \in A | x \ll_c a\}$ is a σ–ideal of A.*

Now we introduce a concept of countably approximating poset.

DEFINITION 1.4. *A poset A with countably directed joins is said to be **countably approximating** if the join map $\vee : \sigma Idl(A) \to A$ $(I \mapsto \vee I)$ has a left adjoint.*

Using countably way below relation, we characterize countably approximating posets.

THEOREM 1.5. *For a poset A with countably directed joins, A is countably approximatingiff for any $a \in A$, $\downarrow_c a$ is a σ–ideal and $a = \vee \downarrow_c a$.*

Proof. Suppose A is countably approximating and let $r : A \to \sigma Idl(A)$ be a left adjoint of $\vee$, i.e., for any $a \in A$ and $I \in \sigma Idl(A)$, $r(a) \subseteq I$ iff $a \leq \vee I$. Take any $b \in r(a)$ and any σ–ideal I with $a \leq \vee I$, $b \in r(a) \subset I$ and hence $b \ll_c a$ i.e., $b \in \downarrow_c a$. Thus we have $r(a) \subseteq \downarrow_c a$. On the other hand, take any $b \in \downarrow_c a$. Since $r(a) \subseteq r(a)$, $a \leq \vee r(a)$; hence $b \in r(a)$. Thus we have $\downarrow_c a \subseteq r(a)$. In all, $\downarrow_c a = r(a)$ is a σ–ideal of A. Since $\downarrow_c a = r(a) \subseteq \downarrow_c a$, $a \leq \vee \downarrow_c a \leq a$, so that $a = \vee \downarrow_c a$.

Conversely, suppose $\downarrow_c a$ is a σ–ideal and $a = \vee \downarrow_c a$ for all $a \in A$, then let $r : A \to \sigma Idl(A)$ be the map defined by $r(a) = \downarrow_c a$. By the above proposition, r is clearly an isotone. Suppose $a \leq \vee I$ for a σ–ideal I of A, then for any $x \in r(a)$, $x \in I$; hence $r(a) \subseteq I$. Furthermore,

$r(a) = \downarrow_c a \subseteq I$ implies that $a = \bigvee r(a) \leq \bigvee I$. Thus r is a left adjoint of $\bigvee : \sigma Idl(A) \to A$. This completes the proof.

REMARK 1.6.

1) Since $\downarrow_{c\_} : A \to \sigma Idl(A)$ $(a \mapsto \downarrow_c a)$ is a left adjoint of $\bigvee : \sigma Idl(A) \to A$, one has the following :

A poset A with countably directed joins is countably approximating iff for any $a \in A$ there is a smallest σ–ideal I with $a \leq \bigvee I$.

2) By 4) of Proposition 1.3, a poset A with countable joins is countably approximating iff for any $a \in A$, $a = \bigvee \downarrow_c a$. Hence every countably approximating lattice (See [9]) is a countably approximating poset. But the converse need not be true. Indeed, take any infinite discrete ordered set A, then $\sigma Idl(A) = \{\{x\} \mid x \in A\}$. Thus $\sigma Idl(A)$ can be identified with A and the join map $\bigvee : \sigma Idl(A) \to A$ is the identity map 1_A, which has clearly a left adjoint $r : A \to \sigma Idl(A)$ $(r(x) = \{x\})$. Hence A is a countably approximating poset but not complete.

DEFINITION 1.7. *A poset A is said to be σ–algebraic if there is a poset X such that A is isomorphic with the σ–ideal poset $\sigma Idl(X)$.*

Suppose $A = \sigma Idl(X)$ for a poset X, then for any $x \in X$, $\downarrow x \ll_c \downarrow x$, because the join of countably directed family in $\sigma Idl(X)$ is given by the union of the family. The converse is also true. Indeed, for any $I \in \sigma Idl(X)$, $\{\downarrow x \mid x \in I\}$ is again countably directed; hence for any $I \ll_c I = \bigvee\{\downarrow x \mid x \in I\} = \bigcup\{\downarrow x \mid x \in I\}$, there is an $x \in I$ with $I \subseteq \downarrow x \subseteq I$ so that $I = \downarrow x$. Using this, one has the following :

PROPOSITION 1.8. *Every σ–algebraic poset is countably approximating.*

Proof. Let $A = \sigma Idl(X)$ for some poset X, then for any $I \in A$, $I = \bigcup\{\downarrow x \mid x \in I\} = \bigvee\{\downarrow x \mid x \in I\} \subseteq \bigvee\{J \in \sigma Idl(X) \mid J \ll_c I\} \subseteq I$ and therefore, $I = \bigvee\{J \mid J \ll_c I\}$. It remains to show that $S = \{J \mid J \ll_c I\}$ is a σ–ideal in A. Clearly, the set S is a lower set. Now take any sequence (J_n) in S, then for any n, $J_n \ll_c I = \bigvee\{\downarrow x \mid x \in I\}$, so that there is an $x_n \in I$ with $J_n \subseteq \downarrow x_n$. Let x_0 be an upper bound of (x_n) in I, then $J_n \subseteq \downarrow x_n \subseteq \downarrow x_0 \ll_c \downarrow x_0 \subseteq I$ so that $\downarrow x_0$ is an upper bound of (J_n) in S.

LEMMA 1.9. *Suppose A is a countably approximating poset and $e : A \to A$ is a self map such that $e \circ e = e$ and e preserves countably directed joins. Then a subposet $e(A)$ of A is again countably approximating.*

Proof. We note that e is an isotone. Let $B = e(A)$. Since any countably directed subset D in B is clearly countably directed in A, the join $\vee_A D$ of D in A exists. Since $e(\vee_A D) = \vee_A e(D)$ and $e \circ e = e$, $e(\vee_A D) = \vee_A D$, so that $\vee_A D$ belongs to B, which is clearly the join $\vee_B D$ of D in B. Hence B has countably directed joins. Suppose $a \ll_c b$ in A and $b \in B$, then for any countably directed subset S of B with $b \leq \vee_B S = \vee_A S$, there is an $s \in S$ with $a \leq s$ and hence $e(a) \leq e(s) = s$. Thus we have $e(a) \ll_c b$ in B. For any $b \in B$, $b = e(b) = e(\vee_A\{a \in A | a \ll_c b \text{ in } A\}) = \vee_A\{e(a) | a \ll_c b \text{ in } A\} = \vee_B\{e(a) | a \ll_c b \text{ in } A\}$, for $\{a \in A | a \ll_c b \text{ in } A\}$ is countably directed. But $b \leq \vee_B\{e(a) | a \ll_c b \text{ in } A\} \leq \vee_B\{x | x \ll_c b \text{ in } B\} \leq b$; therefore $b = \vee_B\{x | x \ll_c b \text{ in } B\}$. Take any sequence (b_n) in $\{x | x \ll_c b \text{ in } B\}$, then there is a_n in A with $b_n \leq e(a_n)$ and $a_n \ll_c b$ in A, for $b = \vee_A\{e(a) | a \ll_c b \text{ in } A\}$. There is a_0 in A such that $a_0 \ll_c b$ in A and $a_n \leq a_0$ for all n. Thus $b_n \leq e(a_n) \leq e(a_0)$ for all n and $e(a_0) \ll_c b$ in B, i.e., $e(a_0)$ is an upper bound of (b_n) in $\{x | x \ll_c b \text{ in } B\}$. This completes the proof.

Using the above lemma, one has the relationship between countably approximating posets and σ–algebraic posets.

THEOREM 1.10. *A poset with countably directed joins is countably approximating iff it is a retract of a σ–algebraic poset by maps preserving countably directed joins.*

Proof. Suppose A is a countably approximating poset. Let $s: \sigma Idl(A) \to A$ be the join map, i.e., $s(I) = \vee I$ and $r : A \to \sigma Idl(A)$ the map defined by $r(x) = \downarrow_c x$. Then for any $x \in A$, $s(r(x)) = \vee \downarrow_c x = x$, i.e., $s \circ r = 1_A$. Furthermore, r and s have right adjoints; hence they preserve joins,in particular, countably directed joins, so that A is a retract of σ–algebraic poset $\sigma Idl(A)$ via r and s.

Conversely, suppose there is a σ–algebraic poset B and there are maps $f : A \to B$ and $g : B \to A$ preserving countably directed joins with $g \circ f = 1_A$. Then let $e : B \to B$ be the map $f \circ g$, then e clearly preserves countably directed joins and $e \circ e = e$. Thus $e(B) = f(g(B)) = f(A)$ is by the above lemma a countably approximating poset and moreover, A and $f(A)$ are isomorphic posets. Thus A is also a countably approximating poset. This completes the proof.

REMARK. *Since a retract of a complete lattice is again complete, one can have similar characterization of countably approximating lattices (See also [9]).*

2. Topological characterization of countably approximating posets

In this section, we define σ_c–topology on a poset and then characterize countably approximating posets and countably approximating lattices by their σ_c–topology.

Let A be a poset and let $\sigma_c(A) = \{U \subseteq A | U = \uparrow U$ and for any countably directed subset S of A with $\vee S \in U$, $S \cap U \neq \phi\}$. Then it is easy to show that $\sigma_c(A)$ is closed under arbitrary unions. Moreover, for any sequence (U_n) in $\sigma_c(A)$, let $U = \cap U_n$, then $U = \uparrow U$. Suppose $\vee S \in U$ for a countably directed subset S of A, then $\vee S \in U_n$ for all n; hence $S \cap U_n \neq \phi$. Pick $s_n \in S \cap U_n$ and let s_0 be an upper bound of (s_n) in S, then $s_0 \in S \cap U$. Thus U is again a member of $\sigma_c(A)$. Thus $\sigma_c(A)$ is a topology on A whose G_δ–sets are again open. We call the topology $\sigma_c(A)$ on a poset A the σ–Scott topology on S. Clearly every Scott open set on A, i.e., a subset U of A which is an upper set and for any directed $S \subseteq A$ with $\vee S \in U$, $U \cap S \neq \phi$, is a σ–Scott open set.

REMARK 2.1. *1) A subset F of a poset S is closed in $(A, \sigma_c(A))$ iff $F = \downarrow F$ and F is closed under the formation of countably directed joins.*
2) Let A, B be posets and $f : A \to B$ a map, then $f : (A, \sigma_c(A)) \to (B, \sigma_c(B))$ is continuous iff f perserves countably directed joins.

THEOREM 2.2. *(Interpolation property) Let A be a countably approximating poset. If $a \ll_c b$ in A, then there is an $x \in A$ such that $a \ll_c x \ll_c b$.*

Proof. Let $S = \{d \in A|$ there is $x \in A$ with $d \ll_c x \ll_c b\}$. Pick any $x \ll_c a$, then $x \ll_c a \ll_c b$ so that $\downarrow_c a \subseteq S$; hence $S \neq \phi$. Take any sequence (d_n) in S. Let x_n be an element of A with $d_n \ll_c x_n \ll_c b$ for all n. Since $\downarrow_c b$ is a σ–ideal, (x_n) has an upper bound, say x_0 in $\downarrow_c b$, so that $d_n \ll_c x_n \leq x_0$. Since $\downarrow_c x_0$ is also a σ–ideal, (d_n) has an upper bound, say d_0 in $\downarrow_c x_0$. Then for all n, $d_n \leq d_0$ and $d_0 \ll_c x_0 \ll_c b$. Hence d_0 is an upper bound of (d_n) in S. Thus S is countably directed. Since $\vee S = \vee\{\vee \downarrow_c x | x \ll_c b\} = \vee \downarrow_c b$ and $a \ll_c b$, there is a $d \in S$ with $a \leq d$. Since $d \in S$, there is an $x \in A$ with $d \ll_c x \ll_c b$; hence we have $a \ll_c x \ll_c b$. This completes the proof.

PROPOSITION 2.3. *For a countably approximating poset A, we have the following:*

1) The family $\{\uparrow_c a | a \in A\}$ forms a base for $\sigma_c(A)$, where $\uparrow_c a = \{x \in A | a \ll_c x\}$.

2) For any $U = \uparrow U$, $\overset{\circ}{U} = \cup\{\uparrow_c b | b \in U\}$, where $\overset{\circ}{U}$ denotes the interior of U in $(A, \sigma_c(A))$.

Proof. 1) For any $a \in A$, $\uparrow_c a$ is an upper set by Proposition 1.3. Suppose $\vee S \in \uparrow_c a$ for some countably directed subset S of A. By Theorem 2.2, there is a $b \in A$ with $a \ll_c b \ll_c \vee S$; hence there is an $s \in S$ with $b \leq s$, so that $s \in \uparrow_c a \cap S$. Thus $\uparrow_c a$ is an open set in $(A, \sigma_c(A))$. Now take any $U \in \sigma_c(A)$ and any $b \in U$. Since $\vee \downarrow_c b = b \in U$ and $\downarrow_c b$ is countably directed, $\downarrow_c b \cap U \neq \phi$. Pick any $a \in \downarrow_c b \cap U$, then $b \in \uparrow_c a \subseteq \uparrow U = U$. Thus $\{\uparrow_c a \,|\, a \in A\}$ forms a base for $\sigma_c(A)$.

2) Since $\cup\{\uparrow_c b \,|\, b \in U\} \subseteq U$, $\cup\{\uparrow_c b \,|\, b \in U\}$ is contained in $\overset{\circ}{U}$. Take any $x \in \overset{\circ}{U}$, then $x = \vee \downarrow_c x \in \overset{\circ}{U}$, so that $\downarrow_c x \cap U \neq \phi$. Pick $a \in \downarrow_c x \cap \overset{\circ}{U}$, then $a \in \overset{\circ}{U} \subseteq U$ and $a \ll_c x$, so that $x \in \uparrow_c a \subseteq \cup\{\uparrow_c b \,|\, b \in U\}$. Thus $\overset{\circ}{U}$ is contained in $\cup\{\uparrow_c b \,|\, b \in U\}$. This completes the proof.

THEOREM 2.4. *Consider the following three statements for a poset A:*

- a) *A is countably approximating.*
- b) *For any $U \in \sigma_c(A)$, $U = \cup\{\uparrow_c x \,|\, x \in U\}$ and for any $x \in A$, $\downarrow_c x$ is a σ–ideal.*
- c) *For any $x \in A$, $x = \vee\{\wedge U | x \in U \in \sigma_c(A)\}$.*

Then one has,

1) *a) and b) are equivalent for any poset A with countably directed joins.*

2) *a), b) and c) are all equivalent for any complete lattice A.*

Proof. 1) a) $\Longrightarrow$ b): This implication is immediate from the above Proposition.

b) $\Longrightarrow$ a): It is enough to show that for any $x \in A$, $x = \vee \downarrow_c x$. Let $y = \vee \downarrow_c x$, then $y \leq x$. Suppose $y \neq x$. Since $\downarrow y$ is closed in $(A, \sigma_c(A))$, $A- \downarrow y$ is an open neighborhood of x. Hence by the assumption, there is a $z \in A- \downarrow y$ with $x \in \uparrow_c z \subseteq A- \downarrow y$. Hence $z \ll_c x$. Thus we have $z \leq y$, which is a contradiction to the fact that $z \in A- \downarrow y$.

2) Suppose A is a complete lattice. It remains to show that a) and c) are equivalent. Suppose A is a countably approximating lattice and $y = \vee\{\wedge U | x \in U \in \sigma_c(A)\}$. Clearly one has $y \leq x$. Suppose $y \neq x$. Then $x \in A- \downarrow y \in \sigma_c(A)$. By the equivalence between a) and b), there is a $z \in A- \downarrow y$ such that $x \in \uparrow_c z \subseteq A- \downarrow y$. By the above proposition, $x \in \uparrow_c z \in \sigma_c(A)$; hence $z \leq \wedge \uparrow_c z \leq y$, which is a contradiction. Conversely, assume c). For any $x \in A$ and $x \in U \in \sigma_c(A)$, we claim $\wedge U \ll_c x$. Indeed, take any σ–ideal I with $x \leq \vee I$. Since $x \in U = \uparrow U$, $\vee I \in U$; hence $I \cap U \neq \phi$. Pick $u \in I \cap U$, then $\wedge U \leq u$ and therefore $\wedge U \in I$. Thus $\wedge U \ll_c x$. Now by the assumption, $x = \vee\{\wedge U | x \in U \in \sigma_c(A)\} \leq \vee \downarrow_c x \leq x$ and therefore, $x = \vee \downarrow_c x$. Thus A is a countably approximating lattice.

REMARK. *It is known [12] that a complete lattice A is continuous iff for any $x \in A$, $x = \vee\{\wedge U | U$ is a Scott–open neighborhood of $x\}$. We define a complete lattice A to be δ–continuous if for any $x \in A$, $x = \vee\{\wedge U | U$ is a G_δ – Scott neighborhood of $x\}$. Then for a complete lattice A, we have the following implication:*

A is continuous $\Longrightarrow$ A is δ-continuous $\Longrightarrow$ A is countably approximating, for $\{U|U$: Scott–neighborhood of $x\} \subseteq \{U|U : G_\delta$–Scott neighborhood of $x\} \subseteq \{U|U : \sigma_c$–neighborhood of $x\}$. We don't yet know anything about the reverse implication.

Another application of Theorem 2.2 is that $\sigma_c(A)$–open filters on a countably approximating poset determine the order structure of A. Indeed, one has the following:

LEMMA 2.5. *Let $a \not\leq b$ in a countably approximating poset A. Then there is a filter F on A such that $F \in \sigma_c(A)$, $a \in F$ and $b \notin F$.*

Proof. Since $a = \bigvee \downarrow_c a \nleq b$, there is an $x_0 \in A$ such that $x_0 \ll_c a$ and $x_0 \nleq b$. By Theorem 2.2, there is a sequence (d_n) such that $x_0 \ll_c d_{n+1} \ll_c d_n \ll_c a$ for all n. Thus let $F = \bigcup_n \uparrow_c d_n = \bigcup_n \uparrow d_n$; then F is a filter, for each $\uparrow d_n$ is a filter and $(\uparrow d_n)_n$ is directed. Moreover, F is open in $(A, \sigma_c(A))$. Since $d_1 \ll_c a$, $a \in F$; since $x_0 \nleq b$, $d_n \nleq b$ for all n, so that $b \notin F$. This completes the proof.

By the above lemma, the following is immediate :

THEOREM 2.6. *Suppose A is a countably approximating poset. Then $a \leq b$ in A iff for any $\sigma_c(A)$–open filter F with $a \in F$, $b \in F$.*

References

1. B. Banachewski, *Hulls, kernels, and continuous lattices*, Houston J. Math. 4 (1978), 517–525.
2. B. Banaschewski and R.–E. Hoffmann (eds.), *Continuous Lattices*, Proceedings of the Conference on Topological and Categorical Aspects of Continuous Lattices, 1979 Lect. Notes in Math. Vol.871, Springer–Verlag, 1981.
3. G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, D.S.Scott, *A Compendium of Continuous Lattices*, Springer–Velgag, 1981.
4. C.R.A. Gilmour, *Real compact–fine Alexandroff Spaces*, Session on Cat. Alg. and Top, Univ. of Cape Town, 65–77.
5. G. Grätzer, *General Lattice Theory*, Birkhäuser, Basel, 1978.
6. R.–E. Hoffmann, *Continuous posets and adjoint sequences*, Semigroup Forum 18 (1979), 173–188.
7. R.–E. Hoffmann, *Continuous posets, prime spectra of completely distributive complete lattices, and Hausdorff compactifications*, Lect. Notes in Math. 871 (1981), Springer–Verlag, 159–208.
8. R.–E. Hoffmann and K.H. Hofmann (eds.), *Continuous lattices and their applications*, Marcel Dekker, Inc., 1985.
9. S.O. Lee, *On countably approximating lattices*, J. Korean Math. Soc. Vol.25, No.1 (1988), 11–23.
10. P.T. Johnstone, *Stone Spaces*, Cambridge Univ. Press, 1982.
11. J. Madden and J. Vermeer, *Lindelöf locales and realcompactness*, Math. Proc. Cambridge Phil., 99(1986), 473–480.
12. D.S. Scott, *Continuous lattices*, Lect. Notes in Math. 274 (1972), Springer–Verlag, 97–136.

Sookmyung Women's University
Seoul 140–742, Korea,

Sogang University
Seoul 121–742, Korea,

Hanyang University
Seoul 133–791, Korea

and

Sung Kyun Kwan University
Seoul 110–745, Korea