Comm. Korean Math. Soc. 4(1989), No. 1, pp. 121~128

CAUCHY PROBLEM FOR CARLEMAN EQUATION BY FINITE DIFFERENCE METHODS

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1. Introduction

We shall consider the coupled system of first order hyperbolic nonlinear equations

$$
\begin{aligned}
u_t + u_x &= v^2 - u^2 \\
v_t - v_x &= u^2 - v^2
\end{aligned}
\tag{1.1}
$$

with initial conditions

$$
u(x,0) = f(x), \quad v(x,0) = g(x), \tag{1.2}
$$

where f and g are bounded measurable functions. System (1.1) is known as the Carleman equation. It was developed to model the spatio–temporal behavior of the velocity distribution function of a gas whose molecules move parallel to the x–axis with constant and equal speed, either in the direction of increasing x or in the direction of decreasing x.

The initial value problem for the Carleman equation has been studied by various analytic methods [2,3,5,6,7]. In [6], Tartar stated the existence and uniqueness of the initial value problem (without proof) with bounded initial data using a fixed point argument. Later, Kaper and Leaf [5] treated the problem in a unified manner and improved some of the previous results by considering the abstract initial value problem associated to the Carleman equation. Decaying property of solutions in time was discussed in [3,5].

Here, we shall prove the uniqueness and existence of a solution for the problem (1.1) and (1.2) for bounded initial data by a finite difference method.

Received December 19, 1988.

2. Main theorem and the proof of uniqueness

DEFINITION 2.1. *A pair of bounded measurable function (u,v) is called a (weak) solution of the initial value problem (1.1) and (1.2), provided that (2.1) and (2.2) hold for all ϕ, where ϕ is a C^1 function which vanishes outside of a compact subset in $R \times [0,\infty)$, that is, $supp\phi \cap \{R \times 0,\infty)\} \subset D$, for some rectangle $D = \{(x,t)|a \le x \le b,\ 0 \le t \le T\}$, so chosen that $\phi = 0$ outside of D and on the lines $t = T$, $x = a$ and $x = b$:*

$$(2.1)\qquad \int_0^\infty \int_{-\infty}^\infty [u(\phi_t + \phi_x) + (v^2 - u^2)\phi]dx\,dt + \int_{-\infty}^\infty f(x)\phi(x,0)dx = 0,$$

$$(2.2)\qquad \int_0^\infty \int_{-\infty}^\infty [v(\phi_t - \phi_x) + (u^2 - v^2)\phi]dx\,dt + \int_{-\infty}^\infty g(x)\phi(x,0)dx = 0.$$

Let C_0^1 be the space of all C^1 functions ϕ such as in definition 2.1. We say that a bounded function f on R is of bounded variation if for any $r > 0$ and any real number y there is a constant $C(r) > 0$ such that

$$\int_{|x|\le r} |f(x+y) - f(x)|dx \le C(r)|y|.$$

THEOREM 2.1. *For any initial data f and g of bounded variation with $0 \le f(x), g(x) \le M$, the initial value problem (1.1) and (1.2) has a unique (weak) solution u and v in $L^\infty(R \times [0,\infty))$ with $0 \le u(x,t), v(x,t) \le M$ on $R \times [0,\infty)$.*

Proof of uniqueness : Let (u_1, v_1) and (u_2, v_2) be 2 sets of the solution as in theorem 2.1, and set $u = u_1 - u_2$ and $v = v_1 - v_2$. Then, we have

$$(2.3)\qquad \int_0^\infty \int_{-\infty}^\infty u(\phi_t + \phi_x) + (v_1^2 - v_2^2 - u_1^2 + u_2^2)dx\,dt = 0,$$

$$\int_0^\infty \int_{-\infty}^\infty v(\phi_t + \phi_x) + (u_1^2 - u_2^2 - v_1^2 + v_2^2) dx\, dt = 0. \tag{2.4}$$

Adding (2.3) and (2.4) gives

$$\int_0^\infty \int_{-\infty}^\infty (u + v)(\phi_t + \phi_x) dx\, dt = 0. \tag{2.5}$$

Since, for any given $\psi \in C_0^1$, we can solve $\phi_t + \phi_x = \psi$ for $\phi \in C_0^1$, $u+v = 0$ a.e. for $t \geq 0$. Then, $u = v = 0$ a.e. for $t \geq 0$ since $u \geq 0$ and $v \geq 0$ for $t \geq 0$.

3. Approximate solutions by a finite difference method

For any fixed $T > 0$ and a positive integer n, let $h = \frac{T}{n}$ and $Q_T = R \times [0, T]$.

First we shall construct an approximate solution on Q_T by using a finite difference scheme.

Define $f_k(x)$ and $g_k(x)$ for $0 \leq k \leq n$ inductively as :

$$f_0(x) = f(x), \quad g_0(x) = g(x) \tag{3.1}$$

$$f_{k+1}(x) = f_k(x - h) + h[g_k^2(x - h) - f_k^2(x - h)] \tag{3.2}$$

$$g_{k+1}(x) = g_k(x + h) + h[f_k^2(x + h) - g_k^2(x + h)]. \tag{3.3}$$

Define approximate solutions $u_n(x, t)$ and $v_n(x, t)$ on Q_T as

$$u_n(x, t) = f_k(x), \quad kh \leq t < (k + 1)h \tag{3.4}$$

$$v_n(x, t) = g_k(x), \quad kh \leq t < (k + 1)h, \quad 0 \leq k \leq n - 1. \tag{3.5}$$

LEMMA 3.1. *If n is so large that $2Mh \leq 1$, then for $0 \leq k \leq n$,*

$$0 \leq f_k(x), g_k(x) \leq M. \tag{3.6}$$

Proof. It follows immediately from (3.2) and (3.3) by induction on k. From now on, we always assume that $2Mh \leq 1$ and that f and g are the same as in Theorem 2.1.

LEMMA 3.2. *The function $f_k(x)$ and $g_k(x)$ are also of bounded variation. Moreover, for each $r > 0$, there is a constant $C(r) > 0$ (independent of k and n) such that*

$$\int_{|x|\leq r} |f_k(x+h) - f_k(x)|dx \leq C(r)h \tag{3.7}$$

$$\int_{|x|\leq r} |g_k(x+h) - g_k(x)|dx \leq C(r)h. \tag{3.8}$$

Proof. For each $r > 0$, let $C_0(r) > 0$ be such that

$$C_0(r) > \max\{\sup_{y\neq 0} \frac{1}{|y|} \int_{|x|\leq r} |f(x+y) - f(x)|dx, \\ \sup_{y\neq 0} \frac{1}{|y|} \int_{|x|\leq r} |g(x+y) - g(x)|dx\}.$$

We claim that for $0 \leq k \leq n$

$$\int_{|x|\leq r} |f_k(x+h) - f_k(x)|dx, \int_{|x|\leq r} |g_k(x+h) - g_k(x)|dx \\ \leq C_0(r)h(1+4Mh)^k. \tag{3.9}$$

When $k = 0$, it is obvious. Assume that it is true up to k. From (3.2) and (3.6), we get

$$|f_{k+1}(x+h) - f_k(x)| \leq |f_k(x) - f_k(x-h)| + 2Mh|g_k(x) - g_k(x-h)| \\ + 2Mh|f_k(x) - f_k(x-h)|$$

from which $(3.9)_{k+1}$ follows immediately for f and so similarly for g. Since $(1+4Mh)^k \leq (1+4MT/n)^n \leq \exp(4MT)$, $0 \leq k \leq n$, we may take $C(r) = C_0(r)\exp(4MT)$.

LEMMA 3.3. *For each $r > 0$, there is a constant $d(r) > 0$ such that*

$$\int_{|x|\le r} |f_{k+1}(x) - f_k(x)|dx \le d(r)h \tag{3.10}$$

$$\int_{|x|\le r} |g_{k+1}(x) - g_k(x)|dx \le d(r)h. \tag{3.11}$$

Proof. For example, (3.10) follows easily from (3.2), (3.6) and (3.7) with $d(r) = C(r) + 4M^2 r$, where $C(r)$ is the constant found in Lemma 3.2.

4. Convergence of approximate solutions

Let us consider the space $L^1_{loc}(R)$ with the metric

$$d(\phi, \psi) = \sum_{j=1}^{\infty} 2^{-j} \frac{\|\phi - \psi\|_j}{1 + \|\phi - \psi\|_j}$$

where $\|\phi\|_j = \int_{|x|\le j} |\phi(x)|dx$ so that a sequence $\{\phi_j(x)\}$ converges to ϕ in $L^1_{loc}(R)$ iff it converges to ϕ in $L^1(K)$ for every compact subset K of R. We need the following elementary facts (cf. [1,4]).

LEMMA 4.1. *If H is a bounded subset of $L^\infty(R)$ satisfying for each $r > 0$*

$$\lim_{h\to 0} \sup_H \int_{|x|\le r} |\phi(x+h) - \phi(x)|dx = 0 \tag{4.1}$$

then H is precompact in $L^1_{loc}(R)$.

LEMMA 4.2. *Let X be a separable space and Y a complete metric space with metric d. If a sequence $\{\phi_j\} : X \to Y$ of continuous functions satisfy*

(a) *for any p in X and $\epsilon > 0$, there is an integer N and a neighborhood $U(p)$ of p such that*

$$d(\phi_j(p), \phi_j(q)) < \epsilon \text{ if } j \ge N \text{ and } q \in U(p)$$

(b) *for any fixed p in X, the set* $\{\phi_j(p) | \ j \geq 1\}$ *is precompact in* Y, *then there is a subsequence of* $\{\phi_j\}$ *which converges to a continuous function on* X *uniformly on any compact subset of* X.

For n large so that $2Mh \leq 1$, *consider the sequence of approximate solutions* $\{u_n(x,t)\}$ *and* $\{v_n(x,t)\}$. *We may view (cf. Lemma 3.3)* $u_n(x,t) = u_n(t)(x)$ *and* $v_n(x,t) = v_n(t)(x)$ *as continuous functions on* $[0,T]$ *valued in* $L^1_{loc}(R)$.

LEMMA 4.3. *The sequences* $\{u_n(t)\}$ *and* $\{v_n(t)\}$ *satisfy the conditions (a) and (b) of Lemma 4.2.*

Proof. (a): For any $\epsilon > 0$, choose N so large that $\sum_{j>N} 2^{-j} < \frac{\epsilon}{2}$. Then by Lemma 3.3, we have

$$\begin{aligned} d(u_n(t), u_n(s)) &\leq \sum_{j=1}^{N} 2^{-j} \|u_n(t) - u_n(s)\|_j + \frac{\epsilon}{2} \\ &\leq \|u_n(t) - u_n(s)\|_N + \frac{\epsilon}{2} \\ &\leq d(N)|t-s| + \frac{\epsilon}{2}. \end{aligned}$$

Hence, it's enough to require $d(N)|t-s| < \epsilon/2$.

(b): For any fixed t in $[0,T]$, $\{u_n(t)\}$ is a bounded subset of $L^\infty(R)$ by (3.4) and (3.6). Hence, by Lemma 4.1, it suffices to show (4.1) for $\{u_n(t)\}$. But it is immediate from (3.7) since $u_n(t)(x) = f_k(x)$ for some k for any fixed t. The proof for $\{v_n(t)\}$ is the same as that of $\{u_n(t)\}$.

Hence, by Lemma 4.2, we may assume that the sequences $\{u_n\}$ and $\{v_n\}$ converge to $u(x,t)$ and $v(x,t)$ in $L^1_{loc}(Q_T)$ respectively and so

$$0 \leq u(x,t),\ v(x,t) \leq M \text{ a.e. in } Q_T$$

by choosing subsequences if necessary.

Since $T > 0$ is arbitrary, the function u and v are in fact defined on $R \times [0,\infty)$ as bounded locally integrable functions.

Finally, we shall prove that the pair (u,v) is a solution of (1.1) and (1.2) although it's quite standard at this moment. Let ϕ be any function in C_0^1 and choose $T > 0$ so large that supp $\phi \cap \{t \geq 0\} \subset Q_T$. Multiply (3.3) by $\phi(x, kh)$ to get

(4.2)

$$f_{k+1}(x)\frac{\phi(x,(k+1)h) - \phi(x,kh)}{h} + f_k(x-h)\frac{\phi(x,kh) - \phi(x-h,kh)}{h}$$
$$+ (g_k^2(x-h) - f_k^2(x-h))\phi(x,kh)$$
$$+ \frac{1}{h}\left[f_k(x-h)\phi(x-h,kh) - f_{k+1}(x)\phi(x,(k+1)h)\right] = 0$$

Multiply (4.2) by h, sum over $k = 0, \cdots, n-1$ and then integrate with respect to x over R.

$$\int_{-\infty}^{\infty}\sum_{k=0}^{n-1} hf_{k+1}(x)\frac{\phi(x,(k+1)h) - \phi(x,kh)}{h}dx \tag{4.3}$$
$$+ \int_{-\infty}^{\infty}\sum_{k=0}^{n-1} hf_k(x-h)\frac{\phi(x,kh) - \phi(x-h,kh)}{h}dx$$
$$+ \int_{-\infty}^{\infty}\sum_{k=0}^{n-1} h(g_k^2(x-h) - f_k^2(x-h))\phi(x,kh)dx$$
$$+ \int_{-\infty}^{\infty}\sum_{k=0}^{n-1}\big[f_k(x-h)\phi(x-h,kh)$$
$$- f_{k+1}(x)\phi(x,(k+1)h)\big]dx = 0$$

Since $\phi(x, nh) = \phi(x, T) = 0$, the last term in (4.3) becomes $\int_{-\infty}^{\infty} f(x)\phi(x,0)dx$. Since ϕ is smooth, $\phi(x,t) = 0$ for $t \geq T$ and $hf_k(x) = \int_{kh}^{(k+1)h} u_n(x,t)dt$, (4.3) may be rewritten as

$$\int_0^{\infty}\int_{-\infty}^{\infty}\big[u_n(\phi_t + \phi_x) + (v_n^2 - u_n^2)\phi\big]\,dx\,dt \tag{4.4}$$
$$+ \int_{-\infty}^{\infty} f(x)\phi(x,0)dx + \delta(h) = 0$$

where $\delta(h) \to 0$ as $h \to 0$, i.e., $n \to \infty$. Noting that the integrations in (4.4) are in fact carried over compact sets, we get (2.1) as $n \to \infty$. Similarly, we can also get (2.2).

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