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ON THE REPRESENTING MEASURE OF A FUNCTION SUBORDINATE TO $(1+z)^2/(1-z)^2$ *

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The function $F(z) = [(1+z)/(1-z)]^2$ maps the unit disc U conformally onto the complex plane minus the negative real axis. A function f is said to be subordinate to F, denoted by $f \prec F$, if $f = F \circ \psi$ for some holomorphic map $\psi : U \to U$ with $\psi(0) = 0$, or equivalently if f maps holomorphically U into the range of F with $f(0) = F(0) = 1$.

We recall the following theorem of Brannan, Clunie and Kirwan [1,2].

THEOREM A. *If $f \prec F$ then there is a unique probability measure μ on the boundary ∂U of U which represents f as*

$$f(z) = \int_0^{2\pi} \left(\frac{1+ze^{-it}}{1-ze^{-it}} \right)^2 d\mu(e^{it}), \quad z \in U. \tag{1}$$

In this short note, we determine the Poisson integral of the representing probability measure μ of $f \prec F$. This gives a method of determing μ from f which we illustrate by examples. An extremal property related to $f \prec F$ is also given as a corollary.

The Poisson integral of a measure μ on ∂U is defined as

$$P[d\mu](z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta - t) + r^2} d\mu(e^{it}), \ z = re^{i\theta}$$

See [3].

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We now prove

THEOREM 1. *If $f \prec F$ is represented by (1), then the Poisson integral $P[d\mu]$ of μ is given by*

$$(2) \qquad P[d\mu](z) = \frac{1}{2\pi} + \frac{1}{4\pi} Re \int_0^r \frac{f(\rho e^{i\theta}) - 1}{\rho} d\rho, \ z = re^{i\theta} \in U.$$

Proof. Let $f(z) = 1 + \sum_1^\infty f_n z^n$. Since

$$\left(\frac{1 + ze^{-it}}{1 - ze^{-it}}\right)^2 = 1 + \sum_1^\infty 4nz^n e^{-int}$$

we have

$$(3) \qquad 1 + \sum_1^\infty f_n z^n = 2\pi\hat{\mu}(0) + \sum_1^\infty 8\pi n\hat{\mu}(n) z^n, \ z \in U,$$

where

$$\hat{\mu}(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{-int} d\mu(e^{it})$$

is the Fourier coefficients of μ. Comparing the corresponding coefficients of (3), we have

$$\hat{\mu}(n) = \begin{cases} 1/2\pi, & n = 0 \\ f_n/8\pi n, & n = 1, 2, \cdots. \end{cases}$$

Since μ is a real measure (in fact, a probability measure) we have

$$\hat{\mu}(-n) = \overline{\mu(n)} = \overline{f_n}/8\pi n, \quad n = 1, 2, \cdots.$$

Therefore

$$\mu \sim \frac{1}{2\pi} + \sum_1^\infty \frac{f_n}{8\pi n} e^{in\theta} + \sum_1^\infty \frac{\overline{f_n}}{8\pi n} e^{-in\theta}.$$

Hence the Poisson integral of μ is given by

$$\begin{aligned}P[d\mu](z) &= \frac{1}{2\pi} + \sum_1^\infty \frac{f_n}{8\pi n} z^n + \sum_1^\infty \frac{\overline{f_n}}{8\pi n} \overline{z}^n \\ &= \frac{1}{2\pi} + \frac{1}{4\pi} \mathrm{Re} \sum_1^\infty \frac{f_n}{n} z^n \\ &= \frac{1}{2\pi} + \frac{1}{4\pi} \mathrm{Re} \int_0^r \frac{f(\rho e^{i\theta}) - 1}{\rho} d\rho, \quad z = re^{i\theta} \in U.\end{aligned}$$

This completes the proof.

COROLLARY 2. *If* $f \prec F$ *then*

$$\frac{4r}{1-r} \geq \mathrm{Re} \int_0^r \frac{f(\rho e^{i\theta}) - 1}{\rho} d\rho \geq -\frac{4r}{1+r}, \quad z = re^{i\theta} \in U. \tag{4}$$

The equality holds on the right or on the left for one value $z_0 = r_0 e^{i\theta_0}$ *if and only if*

$$f(z) = \left(\frac{1 - ze^{-i\theta_0}}{1 + ze^{-i\theta_0}}\right)^2, \quad z \in U$$

or

$$f(z) = \left(\frac{1 + ze^{-i\theta_0}}{1 - ze^{-i\theta_0}}\right)^2, \quad z \in U, \quad \textit{respectively.}$$

Proof. We prove only the lower estimate. The upper estimate can be proved similarly.

If μ is the probability measure on ∂U which represents f by (1), then

$$P[d\mu](z) \geq \frac{1}{2\pi} \frac{1-r}{1+r}, \quad z = re^{i\theta} \in U. \tag{5}$$

Therefore, we have by (2)

$$\mathrm{Re} \int_0^r \frac{f(\rho e^{i\theta}) - 1}{\rho} d\rho \geq 2(\frac{1-r}{1+r} - 1) = \frac{-4r}{1+r}.$$

Now, we note that the equality holds on the right in (4) for $z_0 = re^{i\theta_0}$ if and only if the corresponding equality holds in (5). Since

$$\frac{1-r^2}{1-2r\cos(t-\theta)+r^2} \geq \frac{1-r}{1+r},$$

the equality holds in (5) for $z_0 = r_0 e^{i\theta_0}$ if and only if μ is the unit point mass at $e^{i(\theta_0+\pi)}$. This is the case where

$$f(z) = \left(\frac{1-ze^{-i\theta_0}}{1+ze^{-i\theta_0}}\right)^2.$$

This completes the proof.

Theorem 1 also gives a method of determining the representing measure μ of $f \prec F$. We state it explicitly as a corollary.

COROLLARY 3. *If $f \prec F$, then the representing probability measure μ of f is obtained as the weak limit of*

$$h_r(e^{i\theta}) = \frac{1}{2\pi} + \frac{1}{4\pi} Re \int_0^r \frac{f(\rho e^{i\theta})-1}{\rho} d\rho$$

as $r \to 1$.

Proof is immediate.

We illustrate Corollary 3 by examples.

EXAMPLE 4 $f(z) = \dfrac{1+z}{1-z} \prec F(z)$. We compute

$$\begin{aligned} h_r(e^{i\theta}) &= \frac{1}{2\pi} + \frac{1}{4\pi}\mathrm{Re}\int_0^r \left(\frac{1+\rho e^{i\theta}}{1-\rho e^{i\theta}} - 1\right)\frac{d\rho}{\rho} \\ &= \frac{1}{2\pi} + \frac{1}{2\pi}\mathrm{Re}\int_0^r \frac{e^{i\theta}}{1-\rho e^{i\theta}} d\rho \\ &= \frac{1}{2\pi} - \frac{1}{2\pi}\log|1-re^{i\theta}|, \end{aligned}$$

which converges to $\frac{1}{2\pi}(1-\log 2-\log|\sin\frac{\theta}{2}|)$ as $r\to 1$ in the sense of $L^1(T)$.

Therefore we have

$$\frac{1+z}{1-z}=\int_0^{2\pi}\left(\frac{1+ze^{-it}}{1-ze^{-it}}\right)^2(1-\log 2-\log|\sin\frac{t}{2}|)\frac{dt}{2\pi}.$$

EXAMPLE 5 $f(z)=\left(\frac{1+z^n}{1-z^n}\right)^2\prec F(z),\ n=1,2,\cdots.$

We compute

$$\begin{aligned}2\pi h_r(e^{i\theta})&=1+\frac{1}{2}\operatorname{Re}\int_0^r\left[\left(\frac{1+\rho^n e^{in\theta}}{1-\rho^n e^{in\theta}}\right)^2-1\right]\frac{d\rho}{\rho}\\&=1+2\operatorname{Re}\int_0^r\frac{\rho^{n-1}e^{in\theta}}{(1-\rho^n e^{in\theta})^2}d\rho\\&=1+\frac{2}{n}\operatorname{Re}[\frac{1}{1-r^n e^{in\theta}}-1]\\&=1+\frac{2}{n}\frac{r^n(\cos n\theta-r^n)}{|1-r^n e^{in\theta}|^2}\\&=1+\frac{2}{n}\frac{r^n(1-r^n)}{|1-r^n e^{in\theta}|^2}-\frac{2}{n}\frac{r^n(1-\cos n\theta)}{|1-r^n e^{in\theta}|^2}\\&=1+(I)+(II).\end{aligned}$$

We easily check that

$$\begin{aligned}(II)&=-\frac{2}{n}(\frac{1}{2}-\frac{(1-r^n)^2}{2|1-r^n e^{in\theta}|^2})\\&=-\frac{1}{n}+\frac{1}{n}\frac{(1-r^n)^2}{|1-r^n e^{in\theta}|^2}\end{aligned}$$

converges to $-\frac{1}{n}$ as $r\to 1$ in the sense of $L^1(T)$.

Now, we show that

$$(I)=\frac{2}{n}\cdot\frac{r^n}{1+r^n}\cdot\frac{1-r^{2n}}{|1-r^n e^{in\theta}|^2}$$

converges weakly to

$$\frac{2\pi}{n^2}\sum_{j=0}^{n-1}\delta_{\omega^j}$$

as $r \to 1$, where $\omega = e^{2\pi i/n}$ is the primitive root of unity and δ_{ω^j} is the unit mass concentrated at the point ω^j. In fact, if $g \in C(\partial U)$ then

$$\begin{aligned}\int_0^{2\pi} g(e^{i\theta})\frac{1-r^{2n}}{|1-r^n e^{in\theta}|^2}d\theta &= \int_{-\pi}^{(2n-1)\pi} g(e^{it/n})\frac{1-r^{2n}}{|1-r^n e^{it}|^2}\frac{dt}{n} \\ &= \frac{1}{n}\sum_{j=0}^{n-1}\int_{-\pi}^{\pi} g(e^{ij\pi/n}e^{i\theta})\frac{1-r^{2n}}{|1-r^n e^{i\theta}|^2}d\theta,\end{aligned}$$

which converges to

$$\frac{2\pi}{n}\sum_{j=0}^{n-1} g(e^{ij\pi/n})$$

as $r \to 1$ since the Poisson kernel is an approximate identity [3, Theorem 11.9]. Therefore we have

$$d\mu = \left(1-\frac{1}{n}\right)\frac{d\theta}{2\pi} + \frac{1}{n^2}\sum_{j=0}^{n-1} d\delta_{\omega^j}.$$

Hence we have the following representation of $f(z)$,

$$\left(\frac{1+z^n}{1-z^n}\right)^2 = \frac{n-1}{n} + \frac{1}{n^2}\sum_{j=0}^{n-1}\left(\frac{\omega^j+z}{\omega^j-z}\right)^2,$$

an identity which can also be proved by an elementary calculations.

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