Comm. Korean Math. Soc. 4(1989), No. 1, pp. 103~112

FINITE NORMALIZING EXTENTIONS OF RINGS

DONG SU LEE

1. Introduction

An over ring S of R is called a finite normalizing extension of R if S is a finitely generated R-bimodule whose generator x_i has normalizing property that is ; $Rx_i = x_iR$. We encountered this extension often in algebraic structure for example ; group rings, skew group rings, twisted group rings, crossed products and matrix rings, etc. We call a generating set $\{x_1, x_2, \cdots, x_n\}$ a normalizing base for S. Especially a finite normalizing extension is free if its normalizing base is free that is; $\sum x_i r_i = 0$ implies all $x_i = 0$.

Recall that a ring R is called singular if there exists some x in R such that $\mathrm{ann}_R(x) = \{r \in R \mid rx = 0\}$ is essential left ideal of R. If R is not singular then we call R nonsingular usually. We get the following theorem for nonsingularity.

THEOREM 1.1. *Let S be a free normalizing extension. Then if S is nonsingular, R is also nonsingular.*

Proof. Supposed that R is singular. Then there exists some $r \in R$ such that $\mathrm{ann}_R(r)$ is an essential left ideal of R. We claim that $\mathrm{ann}_S(r)$ is an essential left ideal of S. Let I be any left ideal of S and $s \in I$ where $s = \sum_{i=1}^{n} x_i a_i$. By renumbering normalizing base we get j such that if $k < j$, $a_k \in \mathrm{ann}_R(r)$ and if $k \geq j$, $a_k \notin \mathrm{ann}_R(r)$. Since $\mathrm{ann}_R(r)$ isessential there exists some b_j such that $0 \neq b_j a_j \in \mathrm{ann}_R(r)$. Let $c_j \in R$ such that $c_j x_j = b_j x_j$. Then $0 \neq c_j(\sum_{i=1}^{n} x_i a_i) \in I$ and

Received November 16, 1988.

$c_j(\sum_{i=1}^{n} x_i a_i) = c_j(\sum_{i=j} x_i a_i) + x_j b_j$. Since $b_j a_j r = 0$ and $a_k r = 0$ for $k < j$, $c_j(\sum_{i=1}^{n} x_i a_i)r = c_j(\sum_{i=j+1}^{n} x_i a_i)r$. Let $c_j x_{j+1} = x_{j+1} d_{j+1}$. If $d_{j+1} a_{j+1} \in \text{ann}_R(r)$, by similar method we can choose b_{j+1}, $c_{j+1} \in R$ such that $0 \neq b_{j+1} d_{j+1} a_{j+1} \in \text{ann}_R(r)$ and $c_{j+1} x_{j+1} = x_{j+1} b_{j+1}$, $0 \neq c_{j+1} c_j(\sum_{i=1}^{n} x_i a_i) \in I$ and $c_{j+1} c_j(\sum_{i=1}^{n} x_i a_i)r = c_{j+1} c_j(\sum_{i=j+2}^{n} x_i a_i)r$. Similarly we can find $c_{j+2}, c_{j+3}, \cdots, c_n$ such that $c_n \cdots c_j(\sum_{i=1}^{n} x_i a_i) \in I \cap \text{ann}_S(r)$. Thus $\text{ann}_S(r)$ is an essential left ideal of S. This is contradiction to the nonsingularity of S.

2. Rational extensions and Jacobson radical

Usually an exact sequence $0 \to A \xrightarrow{f} B \to C \to 0$ is said to be rational if for every module D with $f(A) \subset D \subset B$ and every homomorphism $g : D \to B$ the inclusion $f(A) \subset \text{Ker}(g)$ implies that $g = 0$. Let N be a submodule of M such that the exact sequence $0 \to N \xrightarrow{i} M \to M/N \to 0$ is rational. Then we will say that N is a rational submodule of M or M is a rational extension of N. And we know that this definition is equivalent to the fact that is for every $x, y \neq 0 \in M$, there exists some $r \in R$ such that $rx \in N$ and $ry \neq 0$. A module is called rationally complete if it has no proper rational extension. Clearly every rational extension is essential extension. $Z(M)$ is called a singular submodule of M such that $Z(M) = \{m \in M \mid \text{ann}(m) \text{ is essential ideal of } R\}$.

REMARK 2.1. *Let N be a submodule of a module M. If $Z(N) = 0$ and N is essential in M, then M is a rational extension of N.*

Proof. Let $f : A \to M$ where $N \subset A$ and $f(N) = 0$. Suppose that there exists some $k \in A$ such that $f(k) \neq 0$. Then there exists some $r \in R$ such that $rf(k) \in N$ that is $f(rk) \in N$. Since $Z(N) = 0$, there is an $s \in R$ such that $tsf(rk) \neq 0$ for every $t \in R$ because $\text{ann}(f(rk))$ is not essential. Since $Rsrk \cap N = 0$ for N is essential, $f(tsrk) = 0$ for some $t \in R$. This is contradiction. Hence $f = 0$.

Since S is a left R–module we can consider $\mathrm{Hom}_R(S,M)$ for some R–module M. In this case $\mathrm{Hom}_R(S,M)$ is a S–module via $s*f(x)=f(xs)$ for every $s,x\in S$ and $f\in \mathrm{Hom}_R(S,M)$. L.Soeif showed that if N is an essential R–submodule then $\mathrm{Hom}_R(S,N)$ is an essential R–submodule (and consequently an essential S–submodule) of $\mathrm{Hom}_R(S,N)$. Using this proposition and some lemma, we obtain some our results.

LEMMA 2.2. *Let S be a finite free normalizing extension of a ring R and N be an R–module. If $Z_R(N)=0$, then $Z_R(Hom_R(S,N))=0$ (and consequentely $Z_S(Hom_R(S,N))=0$).*

Proof. Suppose that $Z_R(\mathrm{Hom}_R(S,N))\neq 0$. Then there exists some $f\in Z_R(\mathrm{Hom}_R(S,N))$ such that $f(x_i)\neq 0$ for some i for $f\neq 0$ where each x_i is a normalizing base of S. We calaim that for arbiturary $r\in R$ there exist some $s\in R$ such that $srf(x_i)=0$ that is $f(x_i)$ is contained in $Z(N)$. Let $rf(x_i)\neq 0$, then $f(rx_i)\neq 0$ for f is a left R–module homomorphism. Since $rx_i=x_it$ for some $t\in R$, $f(rx_i)=f(x_it)\neq 0$. On the other hand for some $u\in R$, $(ut)*f=0$ and $ut\neq 0$ because $f\in Z_R\mathrm{Hom}_R(S,N))$. Thus $f(x_iut)=0$. But since $x_iu=sx_i$, $x_iut=srx_i$. Hence $srf(x_i)=f(srx_i)=f(x_iut)=0$ and $sr\neq 0$ for $ut\neq 0$. Thus $f(x_i)$ is contained in $Z(N)$. This is impossible for $Z(N)=0$.

The hypotheses of above lemma can be replaced by one that $\mathrm{ann}(x_i)=0$, because the proof of lemma is dependent on the property that sr is nonzero.

PROPOSITION 2.3. (L.SOUEIF). *Let S be a finite normalizing extension of a ring R. Let M be an R–module and N be a submodule of M. If M is an essential extension of N, then $Hom_R(S,M)$ is an essential extension of $Hom_R(S,N)$.*

Proof. See (12).

THEOREM 2.4. *Let S be a finite free normalizing extension of a ring R. Let M be an R–module and N be a submodule of M. If $Z(N)=0$ and M is an essential extension of N, then $Hom_R(S,M)$ is a rational extension of $Hom_R(S,N)$ as R–module (and consequentely as S–module).*

Proof. By lemma 2.2. we get $Z_R(\mathrm{Hom}_R(S,N))=0$. And $\mathrm{Hom}_R(S,M)$ is an essential extension of $\mathrm{Hom}_R(S,N)$ by proposition 2.3. Thus

$\mathrm{Hom}_R(S,M)$ is a rational extension of $\mathrm{Hom}_R(S,N)$ by remark 2.1.

From Proposition 2.3. we have some corrolaries.

COROLLARY 2.5. *Let N be an R–module. Then if $Hom_R(S,N)$ is S–injective, N is R–injective.*

Proof. See (12).

COROLLARY 2.6. *Let N be a R–module and E its injective hull. Then*

(i) *$Hom_R(S,E)$ is the injective hull of the S–module $Hom_R(S,N)$.*
(ii) *$Hom_R(S,N)=0$ if and only if $N=0$.*

Proof. See (12).

Here we modify Corollary 2.5 in quasi–injective case. Quasi–injectiveness of modules is defined by several ways. A well known result of Johnson–Wong states that a module M is quasi–injective if and only if M is a fully invariant submodule of its injective hull that is ; $fM \subset M$ for every $f \in \mathrm{End}_R E$ where E is injective hull of M. Using this we prove the following theorem.

THEOREM 2.7. *If M is an R–module and $Hom_R(S,M)$ is quasi–injective as S–module, then M is quasi–injective as R–module.*

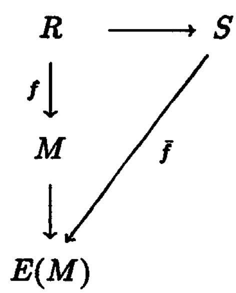
Proof. For arbiturary $f \in \mathrm{End}_R E$ where E is an injective hull of M, we can choose $\overline{f} \in \mathrm{End}_R(\mathrm{Hom}_R(S,E))$ via $\overline{f}(g) = fg$ for every $g \in \mathrm{Hom}_R(S,E)$. Since $\mathrm{Hom}_R(S,E)$ is an injective hull of $\mathrm{Hom}_R(S,M)$ by Corollary 2.7. and $\mathrm{Hom}_R(S,M)$ is quasi–injective as S–module, $\overline{f}(\mathrm{Hom}_R(S,M)) \subset \mathrm{Hom}_R(S,M)$. Thus for every $g \in \mathrm{Hom}_R(S,M)$, $\overline{f}(g)$ is contained in $\mathrm{Hom}_R(S,M)$ that is $fg \in \mathrm{Hom}_R(S,M)$. Thus $fM \subset M$. On the other hand we can prove that above $\overline{f}$ is contained in $\mathrm{End}_R(\mathrm{Hom}_R(S,E))$. At first we know that $fg \in \mathrm{Hom}_R(S,E)$. Secondly $\overline{f}$ is an S–module homomorphism for $\overline{f}(t*g)(x) = \overline{f}(t*g(s)) = f(g(st)) = (fg)(st) = t*(fg)(s) = t*\overline{f}(g(s))$ where $t \in S$.

Recall that the Jacobson radical $J(R) = \{a \in R \mid aM = 0$ for every left irreducible $R-$module $\}$. We reprove that $J(R) = J(S) \cap R$ that was proved by M.Lorenz and R.Resco independently.

PROPOSITION 2.8. (R.RESCO). *Let S be a finite normalizing extension of a ring R. Then $J(R) = J(S) \cap R$.*

Proof. Since every irreducible S–module M is semisimple R–module clealy we obtain $J(R) \subset J(S)$.

For proving $J(S) \cap R \subset J(R)$ it is sufficient to show that for every irreducible R–module M, $aM = 0$ where $a \in J(S) \cap R$. Let M be an irreducible R–module. We can find the injective hull $E(M)$ of M always. Then for arbiturary $f \in \mathrm{Hom}_R(R, M)$ we can find extended homomorphism $\overline{f}$ which is contained in $\mathrm{Hom}_R(S, E(M))$ via the following diagram.



Since M is irreducible $\overline{f}(R) = f(R) = M$ if $\overline{f} \neq 0$. We know that $\mathrm{Hom}_R(R, M)$ and $\mathrm{Hom}_R(S, E(M))$ are R–module and S–module respectively. Let $\Phi : \mathrm{Hom}_R(S, E(M)) \to E(M)$ define by $\Phi(g) = g(1)$. Then Φ is an R–module homomorphism for $\Phi(r * g) = (r * g)(1) = g(r) = rg(1) = r\Phi(g)$. Moreover

$$\mathrm{Ker}(\Phi) = \{g \in \mathrm{Hom}_R(S, E(M)) \mid g(1) = 0\} = 0.$$

Using this R–module homomorphism Φ we can obtain that $R * \overline{f}$ is R–isomorphic to M for $\Phi(R * \overline{f}) = \overline{f}(R) = M$. Since M is irreducible R–module we can know that $\mathrm{Soc}_R(\mathrm{Hom}_R(S, E(M))$ contains $R * \overline{f}$. In this case $S * \overline{f}$ is an artinian R–module for $S * \overline{f}$ contains $R * \overline{f} = M$. Thus $S * \overline{f}$ is an artinian S–module. That implies that $\mathrm{Soc}_S(S * \overline{f}) \neq 0$ for $S * \overline{f}$ contains minimal submodule of itself.

Let $g \in \mathrm{Soc}_S(S * \overline{f}) \subset \mathrm{Soc}_S(\mathrm{Hom}_R(S, E(M))$. Then $R * g$ is isomorphic to $Rg(1)$ as before. Since $Rg(1) \subset E(M)$, $Rg(1) \cap M \neq 0$ for M is essential in $E(M)$. Hence $M \subset Rg(1)$. Therefore we have $M \subset Rg(1) \subset R * g \subset \mathrm{Soc}_S(S * f) \subset \mathrm{Soc}_S(\mathrm{Hom}_R(S, E(M)))$. Since $S * \overline{f}$ is artinian, $\mathrm{Soc}_S(S * \overline{f})$ is semisimple artinian that is ; $\mathrm{Soc}_S(S * \overline{f})$ is a finite direct sum of simple S-modules. For every $a \in J(S) \cap R$, $a(\mathrm{Soc}_S(S * \overline{f}) = 0$. This implies that $aM = 0$ for M is contained in $\mathrm{Soc}_S(S * \overline{f})$.

From this proposition we know that $S/J(S)$ is a finite normalizing extension of $R/J(R)$ for $J(R) = J(S) \cap R$. Thus we get the following corollaries.

COROLLARY 2.9. *If S is a local ring, then R is a local ring.*

Proof. Recall that a ring R is local if and only if $R/J(R)$ is a division ring. Suppose that R is not local ring that is ; $R/J(R)$ is not division ring. Then there exist some proper left ideal K of $R/J(R)$. In fact $(S/J(S))K$ is a proper left ideal of $S/J(S)$ because the fact that $IS = S$ implies that $I = R$ (9).

We call R left perfect if every left R-module M has projective cover. It is well known that R is left perfect if and only if $R/J(R)$ is artinian and $J(R)$ is T-nilpotent.

Also we get the following corollary.

COROLLARY 2.10. *If S is left perfect, so R is.*

Proof. Since $J(R) = R(S) \cap R$ the T-nilpotency of $J(S)$ implies that $J(R)$ is T-nilpotent. And the fact that $S/J(S)$ is artinian implies that $R/J(R)$ is artinian.

3. Strongly primeness and strongly M-primariness

B.S.Chew and J.Negger introduced a generalization of primary ideal. We denote these primarinesses as M-primarinesses in the sense these are defined through R-modules.

DEFINIITON 3.1. *Let R be a ring.*

(1) *An ideal I of R is M-primary ideal if there is a faithful indecomposable R/I-module M.*
(2) *An ideal I of R is strongly M-primary ideal if there is a faithful indecomposable artinian and noetherian R/I-module M.*

They showed that every strongly M–primary ideal is primary in usual sense and every primary ideal is M–primary if R is commutative ring. But the converses are not true. In fact althouth Z is prime, Z is not strongly M–primary for Z has no faithful artinian and noetherian Z–modules. And let $A = F(x, y, z)/ < xy - z^2 >$, then $P = (x^2, xz, z^2)$ is M–primary ideal but not primary ideal. By simple calculation we know· that A/P is a fithful indecomposable A/P–module.

On the other hand D.Handelman and J.Lawrence defined strongly primeness of a ring. We know that some mathematicians call R–strongly prime if $ab = 0$ implies $a = 0$ or $b = 0$. But the concept of their strongly primeness is different and weaker than usual concept.

At first they defined a (left) insulator for $r \in R \setminus\{0\}$ to be a finite subset of R, denoted by $S(r)$ such that

$$\mathrm{ann}_1(sr \mid s \in S(r)) = 0.$$

DEFINITION 3.2. *R is (left) strongly prime if each nonzero element of R has a left insulator. That is, for every $r \in R$; there exists a finite set $S(r)$ such that for $t \in R$, $\{tsr \mid s \in S(r)\} = 0$ implies $t = 0$.*

They showed that left strongly primeness and right strongly primeness are not symmetric by examples (5). But if R is (left or right) strongly prime, R is prime.

In this section we study some properties of strongly primeness and strongly M–primariness respectively. And in finite normalizing case we obtain some results between them.

THEOREM 3.3. *Ler R–be a commutative ring with 1. If R is subdirectly irreducible ring satisfying either chain conditions on ideals, then R is a strongly M–primay ring.*

Proof. We prove that R itself is a faithful indecomposable artinian and noetherian R–module. At first R is a faithful indecomposable R–module for R is subdirectly irreducible. Since every artinian ring is noetherian, artinian implies noetherian. We assume that R is noetherian. Let $N = \mathrm{ann}(hR)$ where hR is the heart of R. Since N is maximal R/N is a field and for some k, $N^k = 0$ by Levizki's Theorem. Then each factor N^{i-1}/N^i is R/N–module with $(x+N^i)(r+N) = xr + N^i$ for $x \in N^{i-1}$ and $r \in R$. Thus N^{i-1}/N^i is a finite dimensional vector space over R/N. This implies that N^{i-1}/N^i has a composition series. Thus we obtain a composition series $R \supset N = N_{1,1} \supset N_{1,2} \cdots \supset N_{1,j} = N^2 = N_{2,1} \supset N_{2,2} \cdots \supset N_{k-1,j} = N^k = 0$. Thus R is artinian.

But left subdirectly irreducibility of R implies that R is not indecomposable left module. Thus every left subdirectly irreducible ring is left M–primary.

Generally primeness of a ring R does not imply strongly M–primariness of R even if R is an integral domain for example an integer ring Z. But with additional condition that is true.

THEOREM 3.4. *If R is strongly prime and left artinian, then R is strongly M-primary.*

Proof. Since R is artinian there exists a minimal right ideal L. We claim that L is an indecomposable fithful artinian and noetherian. Since L is minimal we know that L is both artinian and noetherian indecomposable. On the other hand every element of L has an insulator. Thus $\mathrm{ann}(L) = 0$ that is L is a faithful R–module.

Generally prime ring R is not strongly prime. D.Handleman and J.Lawrence proved the following proposition.

PROPOSITION 3.5. *If R is prime and satisfies the decending chain condition on left (or right) annihilators, then R is left (right) strongly prime.*

Proof. See (5).

From above proposition we know that if R is prime and artinian, then R is strongly M–primary for if R is prime and artinian, then R is strongly prime. And we can get the following.

THEOREM 3.6. *If R is a semiprime left Goldie ring then every essential left ideal of R contains an insulator that is ; there exist some finite set $a_1, a_2, \cdots, a_k \in I$ such that $\mathrm{ann}\{a_i \mid 1 \le i \le k\} = 0$ for every left essential ideal I of R.*

Proof. Since R is semiprime left Goldie, there exist finite minimal prime ideals $P_1, P_2, \cdots P_m$. By proposition, each R/P_i is left strongly prime that is ; each P_i is left strongly prime ideal. Let I be an essential ideal of R. Then I is not contained in any P_i for $I \cap (\cap_{i \neq j} P) \neq 0$ (in fact $\cap_{i=1}^{m} P = 0$). So there exist finite sets $a_{ij} \in I$ such that $\{a_{ij} \mid 1 \leq j \leq i_1\}$ is an insulator of $I + P_i/P_i$ in R/P_i that is $xa_{ij} \in P_i$ for all j implies that $x \in P_i$. let $F = \cup\{a_{ij}\}$. Then F is an insulator of I for $Fx = 0 = \cap_{i=1}^{m} P_i$ implies that $x \in \cap_{i=1}^{m} P_i = 0$.

For the study of finite normalizing extensions, we get the folowing.

THEOREM 3.7. *Let S be a liberal extension of a ring R. If S is strongly prime then R is strongly prime.*

Proof. Since S is strongly prime and every element of R is also contained in R, every $a \in R$ has an insulator $S(a)$ in S. Let $S(a) = \{s_j \mid 1 \leq j \leq t\}$ where $s_j = \sum_{i=1}^{n} r_{ji} x_i$ for some $r_{ji} \in R$, we claim that $\{r_{ji} \mid 1 \leq i \leq n,\ 1 \leq j \leq t\}$ is an insulator for a in R. If $tr_{ji}a = 0$ for every i and j, then for all j, $ts_j a = t(\sum_{i=1}^{n} r_{ji} x_i)a = \sum_{i=1}^{n} (tr_{ji}a)x = 0$ for S is liberal extension of R. Thus $t = 0$.

References

1. L.Bican, T.Kepptk, and P.Nemec, *Rings, Modules and preradicals*, Marcel Dekker, Inc. (1982).
2. B.S. Chwe and J. Negger, *Primary ideals*, J. Korean Math. Soc. 21, No.2 (1984), 141–146.
3. E.Formanek and A.V. Jategaonkar, *Subrings of noetherian rings*, Proc. Amer. Math. Soc. 46 (1974), 181–186.
4. D. Handelman and L. Lawrence, *Strongly prime rings*, Tran. Amer. Math. Soc. 211 (1975), 209–233.
5. D. Handelman, *Strongly semiprime rings*, Pacific Math. 60 (1975), 115–122.
6. A.G. Heinicke and J.C. Robson, *Normalizing Extensions : Prime ideals and incomparability*, J. of Algebra, 72 (1981), 237–268.
7. C. Lanski, *Goldie conditions in finite normalizing extensions*, Proc. Amer. Math. Soc. 79 (1980), 515–519.

8. M. Lorenz and D.S. Passman, *Integrability and normalizing extensions*, J. of Algebra, 61 (1979), 289–297.
9. M. Lorenz and D.S. Passman, *Prime ideals in crossed products of finite groups*, Israel J. Math. 33 (1979), 89–132.
10. M. Lorenz, *Finite normalizing extensions of rings*, Math. Z. 176 (1981), 89–243.
11. R. Resco, *Radicals of finite normalizing extensions*, Comm. Algebra, 7 (1981), 713–725.
12. L. Soueif, *Normalizing extensions and injective modules, essentially bounded normalizing extensions*, Comm. Algebra, 15, No.8, (1987), 1607–1619.

Ulsan University
Ulsan 680–749, Korea