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SMOOTHNESS AND WEAK ASPLUND SPACE *

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1. Introduction

A real Banach space X is said to be an Asplund (respectively, weak Asplund) space if every continuous convex function defined on an open convex subset of X is Fréchet (respectively, Gateaux) differentiable on a dense G_δ subset of its domain. In 1968, Asplund [1] called Asplund space (AS) a strong differentiability space, while he called weak Asplund space (WAS) a weak differentiability space. Asplund proved that every reflexive Banach space admitting a Fréchet differentiable norm is an AS [1]. Latter it was known that every reflexive Banach space is an AS [7], and in 1976, I. Ekeland and G. Lebourg [3] essentially showed that a Banach space admitting a Fréchet differentiable norm is an AS [see also 4,p.170]. At this point it was asked whether latter type of property holds for WAS ; that is, if a Banach space X admits an equivallent smooth norm (smooth norm is the one which is Gateaux differentiable at every point of X except 0), is X a WAS? In contrast to AS, the knowledge of WAS is rather incomplete. So far WAS was characterized by means of the separability of the space [6], rotundity of the dual space of a Banach space, and hence by the subspace of weakly compactly generated space [1]. In 1979, D.G. Larman and R.R. Phelps asked whether every Gateaux differentiability space (GDS) is a WAS and whether the existance of an equivalent smooth norm on the space is either necessary or sufficient for the space to be a WAS, but still these questions are unanswered [5]. Several attempts were made to characterize it by smoothness of the space. In 1987, J.M. Borwein and D. Preiss [2] showed if X is a Banach space with a smooth renorm, then X is a GDS, which is a little short of WAS.

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This paper introduces a particular positive sublinear functional in Lemma 2 and by using it, it shows that if a Banach space X admits an equivalent smooth norm so that in the dual X^* of X, the map $f \to \|f\|$ is weak* upper semi–continuous, then X is a WAS.

DEFINITION. *A real function ϕ on an open convex subset A of a Banach space X is said to be Gateaux differentiable at $x \in A$ in the direction $y \in X$ if*

$$\lim_{t\to 0} \frac{\phi(x+ty)-\phi(x)}{t} \text{ exists.}$$

If the limit exists for every direction $y \in X$, then we call ϕ *Gateaux differentiable at x.*

A real function ψ on a linear space X is said to be positively homogeneous if

$$\psi(tx) = t\psi(x) \quad \text{for all} \quad t > 0 \quad \text{and} \quad x \in X.$$

If ψ is positively homogeneous and convex, it is called *sublinear functional.* The real function f on X which satisfies

$$f(y-x) \leq \phi(y) - \phi(x) \quad \text{for all} \quad y \in A$$

is called a *subgradient of ϕ at x* and we denote the set of all such subgradients by $\partial\phi(x)$. If ϕ is continuous at x, then every $f \in \partial\phi(x)$ is continuous linear functional on X [4, p.127, Theorem 7]. A Banach space X is said to be a *Gateaux differentiability space* if every continuous convex function defined on an open convex subset of X is Gateaux differentiable on a dense (not necessarily a dense G_δ) subset of its domain.

2. Smoothness and WAS

We will use the following theorem.

THEOREM 1. [4, P.134]. *(weak* upper semi–continuity of subgradient mapping)*

In a normed linear space X, for a continuous convex function ϕ defined on an open convex subset A of X, given $x \in A$ and a sequence $\{x_n\}$ in A

where $\|x_n - x\| \to 0$, we have for any sequence $\{f_n\}$ where $f_n \in \partial\phi(x_n)$ that $\{f_n\}$ has a weak* cluster point and all such cluster points are in $\partial\phi(x)$.

LEMMA 2. *Let ϕ be a continuous convex function defined on an open convex subset A of a Banach space X. Define for each $x \in A$, σ_x on X by*

$$\sigma_x(y) = \sup\{f(y) : f \in \partial\phi(x)\},\ y \in X, \quad \text{and let}$$
$$P_x(y) = \sigma_x(y) + \sigma_x(-y).$$

Also define ψ on A by

$$\psi(x) = \sup\{\|f - g\|,\ f, g \in \partial\phi(x)\}.$$

Then

(i) *P_x is a positive sublinear functional on X, and for any fixed $y_0 \in X$, $\{x \in A : P_x(y_0) = 0\}$ is a dense G_δ subset of A.*
(ii) *$\psi(x) = \sup_{\|v\|=1} P_x(v)$.*
(iii) *ψ is upper semi-continuous if $f \to \|f\|$ is weak* upper semi-continuous in X^*, and*
(iv) *$\psi(x) = 0$ if and only if ϕ is Gateaux differentiable at x.*

Proof. (i) By definition of P_x, it is clear that P_x is a positive sublinear functional. For the second part, first we would like to show for a fixed $y_0 \in X$, the map $x \to P_x(y_0)$ is upper semi-continuous. It is sufficient to show that $x \to \sigma_x(y_0)$ is upper semi-continuous. Suppose $x_n \to x$. Then

$$\limsup_{n\to\infty} \sigma_{x_n}(y_0) = \limsup_{n\to\infty} f_n(y_0) \text{ for some } f_n \in \partial\phi(x_n),$$

and by Theorem 1, all weak* cluster points of $\{f_n\}$ are in $\partial\phi(x)$. Hence

$$\limsup_{n\to\infty} \sigma_{x_n}(y_0) \le \sigma_x(y_0).$$

By definition of σ_x,

$$f(-y_0) \le \sigma_x(-y_0),$$

and

$$-\sigma_x(-y_0) \le f(y_0) \le \sigma_x(y_0)$$

for all $f \in \partial\phi(x)$. Hence if $P_x(y_0) = 0$, then $f(y_0)$ has the same value as $\sigma_x(y_0)$ and $-\sigma_x(-y_0)$ for all $f \in \partial\phi(x)$. Therefore given $x_n \to x$ in A,

$$\lim_{n\to\infty} \sigma_{x_n}(y_0) = \lim_{n\to\infty} f_n(y_0) = f(y_0) = \sigma_x(y_0)$$

for some $f_n \in \partial\phi(x_n)$, and $f \in \partial\phi(x)$. Hence

(*) $\{x \in A : P_x(y_0) = 0\} \subseteq \{x \in A : x \to \sigma_x(y_0) \text{ is continuous at } x\}$.

On the other hand, suppose $x \to \sigma_x(y_0)$ is continuous at x_0. For any $t > 0$,

$$f_{x_0}(ty_0) \le \phi(x_0 + ty_0) - \phi(x_0),$$

and

$$-f_{x_0+ty_0}(ty_0) \le \phi(x_0) - \phi(x_0 + ty_0)$$

for all $f_{x_0} \in \partial\phi(x_0)$ and $f_{x_0+ty_0} \in \partial\phi(x_0 + ty_0)$. This implies

$$f_{x_0}(y_0) \le \frac{\phi(x_0 + ty_0) - \phi(x_0)}{t} \le f_{x_0+ty_0}(y_0),$$

and hence

$$\sigma_{x_0}(y_0) \le \frac{\phi(x_0 + ty_0) - \phi(x_0)}{t} \le \sigma_{x_0+ty_0}(y_0).$$

Likewise, we get

$$\begin{aligned}\sigma_{x_0-ty_0}(y_0) &\le \frac{\phi(x_0 - ty_0) - \phi(x_0)}{-t} \\ &\le -\sigma_{x_0}(-y_0), \qquad t > 0.\end{aligned}$$

Since $-\sigma_{x_0}(-y_0) \le \sigma_{x_0}(y_0)$ always, if $x \to \sigma_x(y_0)$ is continuous at x_0, $P_{x_0}(y_0) = 0$ follows from the last two inequalities. This reverses the inclusion in (*). Hence, the conclusion follows because the set of points at which an upper semicontinuous function is continuous is a dense G_δ set [4, p.109].

(ii) There exist $f_n, g_n, \tilde{f}_n, \tilde{g}_n \in \partial\phi(x)$, and $\|v_n\| = 1,\ \|w_n\| = 1$ such that

$$\begin{aligned}
\sup_{\|v\|=1} P_x(v) &= \sup_{\|v\|=1} (\sigma_x(v) + \sigma_x(-v)) \\
&= \lim_{n\to\infty} (\sigma_x(v_n) + \sigma_x(-v_n)) \\
&= \lim_{n\to\infty} (f_n - g_n)(v_n) \\
&\le \sup\{\|f - g\| \,:\, f, g \in \partial\phi(x)\} \\
&= \lim_{n\to\infty} (\tilde{f}_n - \tilde{g}_n)(w_n) \\
&\le \lim_{n\to\infty} (\sigma_x(w_n) + \sigma_x(-w_n)) \\
&= \lim_{n\to\infty} P_x(w_n) \\
&\le \sup_{\|v\|=1} P_x(v).
\end{aligned}$$

(iii) Suppose $x_n \to x$. Then

$$\begin{aligned}
\limsup_{n\to\infty} \psi(x_n) &= \limsup_{n\to\infty} \{\|f_n - g_n\| \,:\, f_n, g_n \in \partial\phi(x_n)\} \\
&= \lim_{k\to\infty} (f_{n_k} - g_{g_k})(v_k)
\end{aligned}$$

for some $f_{n_k}, g_{n_k} \in \partial\phi(x_{n_k})$ and $\|v_k\| = 1$. By taking further subsequence of $\{f_{n_k} - g_{n_k}\}$, we get a weak* limit of the subsequence. We assume $\{f_{n_k} - g_{n_k}\}$ is such a subsequence which weak* converges to $f - g \in (\partial\phi(x) - \partial\phi(x))$. Then

$$\begin{aligned}
\lim_{k\to\infty} (f_{n_k} - g_{n_k})(v_k) &\le \lim_{k\to\infty} \|f_{n_k} - g_{n_k}\| \\
&\le \|f - g\| \le \psi(x).
\end{aligned}$$

(iv) By the definition of ψ, $\psi(x) = 0$ if and only if $\partial\phi(x)$ is a singleton set. Hence ϕ is Gateaux differentiable at x.

Following theorem is due to J.M. Borwein and D. Preiss.

THEOREM 3. [2]. *Let X be a Banach space with a smooth renorm and let $f : U \subset X \to R$ be a convex and continuous on the open U. Then f is densely Gateaux differentiable.*

THEOREM 4. *If a Banach space X admits an equivalent smooth norm so that in the dual X^* of X, the map $f \to \|f\|$ is weak* upper semi-continuous, then X is a WAS.*

Proof. Let X be a Banach space with such renorm, and let ϕ is a continuous convex function defined on an open convex subset A of X. Let ψ on A is defined as in Lemma 2. Since ψ is upper semi-continuous

$$D = \{x \in A : \psi \text{ is continuous at } x\}$$

is a dense G_δ subset of A. We claim that $\psi(x) = 0$ for all $x \in D$. If $\psi(x) > 0$ for some $x \in D$, then since ψ is continuous at x, there exists $\delta > 0$ such that $\psi(y) > 0$ for all $\|y - x\| < \delta$. But ϕ is densely Gateaux differentiable on A by Theorem 3. Hence there exists x_0 at which ϕ is Gateaux differentiable and x_0 is in the δ-neighbourhood of x. This implies $\psi(x_0) = 0$, which is contradiction.

We will give one application of Lemma 2.

THEOREM 5. [MAZUR]. *Every separable Banach space is a WAS.*

Proof. Let X be the given separable Banach space, and let any continuous convex function ϕ and corresponding σ_x be defined as in Lemma 2. Suppose $\{x_n\}$ is a dense generating set of X. Let

$$D_n = \{x \in A : P_x(x_n) = 0\} \quad \text{and} \quad D = \bigcap_{n=1}^{\infty} D_n.$$

Then D is a dense G_δ subset of A. If $x \in D$, then $P_x(x_n) = 0$ for all n, and since P_x is a continuous sublinear functional, we can assume that the set $\{x_n\}$ consists of dense boundary points of the unit ball of X. Therefore for any boundary point v, $P_x(v) = 0$, which implies $\psi(x) = 0$. Hence X is a WAS.

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