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THE CENTRALIZER OF KHα IN KGα *

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1. Introduction

In [2], G.Karpilovsky investigated a K-basis of the center of a twisted group algebra K^tG by utilizing the Berman-Reynolds permutation lemma which can not be generalized to rings in a suitable way.

It is well-known that twisted group algebras are the algebras obtained from group algebras by identifying a central subgroup with a subgroup of the commutative ring's multiplicative group. A very satisfactory characterization of the center of the algebras obtained from group algebras over an integral domain by such identifications has been obtained by A.A. Iskander [1]. Using analogue methods, we shall characterize a K-basis of the centralizer of $KH\alpha$ in $KG\alpha$.

Unless otherwise stated, G will denote a not necessarily finite group, $\zeta(G)$ the center of G, K a commutative ring with 1, and K^* the group of units of K. Let A be an algebra, B a subalgebra of A. The centralizer of B in A will be denote by $C_A(B)$. If $A = B$, then $C_A(A)$, written $\zeta(A)$, is the center of A.

2. Main Results

We let K^tG denote a twisted group algebra of G over K. That is, K^tG is an associative K-algebra which is a free K-module with basis $\{\overline{x} \mid x \in G\}$ and which satisfies the condition that for all $x, y \in G$, $\overline{xy} = r(x,y)\overline{x}\,\overline{y}$, $r(x,y) \in K^*$.

The associativity condition is equivalent to $\overline{x}(\overline{y}\,\overline{z}) = (\overline{x}\,\overline{y})\overline{z}$ for all $x, y, z \in G$ and this is equivalent to $r(x,yz)r(y,z) = r(x,y)r(xy,z)$. If

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$r(x,y) = 1$ for all $x, y \in G$ then K^tG is in fact KG, the group algebra of G over K.

Let N be a central subgroup of a group G and let $\alpha : N \to K^*$ be a homomorphism. If $I(\alpha)$ is the ideal of KG generated by $\{n - \alpha(n)1 \mid n \in N\}$, then we let $KG\alpha$ denote $KG/I(\alpha)$. In fact, $KG\alpha$ is the algebra obtained from the group algebra KG by identifying n with $\alpha(n)$ for every $n \in N$.

We next provide an important link between twisted group algebras and ordinary group algebras. That is, twisted group algebras are $KG\alpha$ for appropriate G and α.

THEOREM 1 [1]. *Let N be a central subgroup of G and let $\alpha : N \to K^*$ be a homomorphism. Then $KG\alpha$ is a twisted group algebra of G/N over K. Furthermore, if B is a transversal of G modulo N, then every element of KG is uniquely writable as a K-linear combination in B plus an element of $I(\alpha)$.*

THEOREM 2. [1]. *Suppose N and α are as in Theorem 1. If L is the kernel of α and α' is the injective homomorphism of N/L into K^* induced by α, then $KG\alpha \cong K(G/L)\alpha'$.*

Let H be a subgroup of G. Then two elements $x, y \in G$ are called H-conjugate if $y = h^{-1}xh$ for some $h \in H$.

It is clear that the H-conjugacy is an equivalence relation and so G is a union of H-conjugacy classes. For a given $g \in G$, let $Cl_H(g)$ denote the H-conjugacy class of g, i.e., $Cl_H(g) = \{h^{-1}gh \mid h \in H\}$. If $G = H$, then $Cl_G(g)$, written $Cl(g)$, is the conjugacy class of g.

LEMMA 3. *Suppose H is a subgroup of G, N is a central subgroup of G such that $N \subseteq H$, and $g \in G$. Let $T_H(g) = [g, H] \cap N$, where $[g, H] = \{g^{-1}h^{-1}gh \mid h \in H\}$. Then*

(1) *$T_H(g) = \{n \in N \mid ng$ is a H-conjugate of $g\}$*
(2) *$T_H(g)$ is a subgroup of N*
(3) *if g is a H-conjugate of g', then $T_H(g) = T_H(g')$*

Proof. Since (1) is obvious, we prove (2) and (3). Let $m, n \in T_H(g)$. Then there are $h, k \in H$ such that $h^{-1}gh = mg$, $k^{-1}gk = ng$. Hence

$(k^{-1}h)^{-1}g(k^{-1}h) = h^{-1}(kgk^{-1})h = h^{-1}n^{-1}gh = n^{-1}h^{-1}gh = n^{-1}mg$. Thus $T_H(g)$ is a subgroup of N. If g is a H–conjugate of g' and $m \in T_H(g)$, there are $h, k \in H$ such that $h^{-1}gh = mg$, $k^{-1}gk = g'$. Hence $k^{-1}(mg)k = mg'$. Thus g, g', mg, mg' are all H–conjugates, i.e., g' is a H–conjugate of mg'. Therefore $T_H(g) \subset T_H(g')$. By symmetry, $T(g) = T(g')$.

We may identify $KH\alpha$ as a subalgebra of $KG\alpha$.

The following theorem describes the centralizer of $KG\alpha$ in $KG\alpha$.

THEOREM 4. *Suppose K is an integral domain, H is a subgroup of G, N is a central subgroup of G such that $N \subseteq H$, and $\alpha : N \to K^*$ is an injective homomorphism. Let $D \subseteq G$ and $\{N_d \mid d \in D\}$ be a transversal for the H/N–conjugacy classes of G/N. Then the set $\{\sum Cl_H(d) + I(\alpha) \mid d \in D,\ |Cl_H(d)| < \infty$ and $T_H(d) = \{1\}\}$ is a basis for $C_{KG\alpha}(KH\alpha)$ as a K–module.*

Proof. **Step** 1. We construct a transversal B of G modulo N such that $D \subseteq B \subseteq \cup\{Cl_H(d) \mid d \in D\}$

For every $d \in D$, let $B(d) \subseteq Cl_H(d)$ be a transversal of $NCl_H(d)$ modulo N such that $d \in B(d)$. Then $B = \cup\{B(d) \mid d \in D\}$ is a transversal of G modulo N

Indeed, let $b \in B(d)$, $b' \in B(d')$ and $N_b = N_{b'}$. Since $B(d) \subseteq Cl_H(d')$, b is a H–conjugate of d and b' is a H-conjugate of d'. Thus N_b is a H/N–conjugate of N_d and $N_{b'}$ is a H/N–conjugate of $N_{d'}$. Hence N_d is a H/N–conjugate of $N_{d'}$ and so $d = d'$. Therefore $b, b' \in B(d)$ and $N_b = N_{b'}$. By the choice of $B(d)$, $b = b'$. Let $g \in G$, then N_g is a H/N–conjugate of N_d in G/N for some $d \in D$. Thus $N_g = N_{h^{-1}}N_dN_h = N_{h^{-1}dh}$ for some $h \in H$ and so $ng = h^{-1}dh$ for some $n \in N$. Hence $ng \in Cl_H(d)$. Therefore there is $b \in B(d)$ such that $N_b = N_{ng} = N_g$.

Step 2. Let $B' = \{b \in B \mid T_H(b) = [b, H] \cap N = \{1\}\}$. Then B' is closed under H–conjugates.

Indeed, if $b \in B'$, then $b \in B(d) \subseteq Cl_H(d)$ and $T_H(b) = 1$. Thus $Cl_H(b) = Cl_H(d)$ and so $T(d) = 1$. If $c \in Cl_H(b) = Cl_H(d)$, $N_c = N_{b'}$ for some $b' \in B(d)$. Hence $c = nb'$ for some $n \in N$. Since $b, c, b' \in Cl_H(d)$, they are all H–conjugates. Thus $n \in T_H(b') = T_H(d) = \{1\}$ and so $n = 1$ or $c = b' \in B(d)$. Hence $B(d) = Cl_H(d)$. Thus the mapping $b \to h^{-1}bh$ is a permutation of B' for every $h \in H$.

Step 3. Suppose $b, c \in B$, $m, n \in N$, $h \in H$ and $h^{-1}bh = mb$, $h^{-1}ch = nb$. Then $b = c$.

Indeed, $h^{-1}m^{-1}nbh = m^{-1}nh^{-1}bh = m^{-1}nmb = h^{-1}ch$. Thus $m^{-1}nb = c$ and so $b = c$.

Step 4. We claim that $C_{KG\alpha}(KH\alpha)$ is generated by $\{\sum Cl_H(d) + I(\alpha) \mid d \in D,\ Cl_H(d) < \infty$ and $T_H(d) = 1\}$ as a K-module.

Let $A = \sum\{t_b b \mid b \in B\} + I(\alpha) \in C_{KG\alpha}(KH\alpha)$. Then $A = h^{-1}Ah$ for all $h \in H$, i.e., $A = \sum\{t_b h^{-1}bh \mid b \in B\} + I(\alpha)$.

Let $c \in B$ and $n \in T_H(c)$. Then $k^{-1}ck = nc$ for some $k \in H$. Thus $I(\alpha) = A - k^{-1}Ak = t_c(1 - \alpha(n))c + \sum\{u_b b \mid b \in B,\ b \neq c\} + I(\alpha)$. By Theorem 1, $t_c(1 - \alpha(n)) = 0$ in K. Since K is an integral domain, $t_c = 0$ or $\alpha(n) = 1$. Since α is injective, $t_c = 0$ or $n = 1$. Hence $A = \sum\{t_b b \mid b \in B'\} + I(\alpha)$. If $c, c' \in B'$ are H-conjugate, then $k^{-1}ck = c'$ for some $k \in H$. Hence $I(\alpha) = A - k^{-1}Ak = (t_{c'} - t_c)c' + \sum\{s_b b \mid b \in B',\ b \neq c'\}$.

By Theorem 1, $t_{c'} = t_c$.

If $c \in B'$ has an infinite conjugacy class, then $t_c = 0$ since only finitely many $t_b \neq 0$. Thus every element of $C_{KG\alpha}(KH\alpha)$ is uniquely writable as $\sum\{t_b b \mid b \in B',\ |Cl_H(b)| < \infty\} + I(\alpha)$, where $t_b \neq 0$ only for a finite number of $b \in B'$ and $t_b = t_c$ if b is a conjugate of c. Conversely, every element of this form belongs to $C_{KG\alpha}(KH\alpha)$ since the subset of B' of all elements with finitely many conjugates is invariant under H-conjugation. Thus $C_{KG\alpha}(KH\alpha)$ is generated by $\{\sum Cl_H(b) + I(\alpha) \mid b \in B',\ |Cl_H(b)| < \infty$ and $T_H(b) = \{1\}\}$ as a K-module. If $b \in B$, $b \in B(d)$ and $T_H(b) = \{1\}$, then $B(d) = Cl_H(d) = Cl_H(b)$. Hence $C_{KG\alpha}(KH\alpha)$ is generated by $\{\sum Cl_H(d) + I(\alpha) \mid d \in D,\ |Cl_H(d)| < \infty$ and $T_H(d) = \{1\}\}$.

Step 5. If $d \neq d'$ and $T_H(d) = T_H(d') = 1$, then $Cl_H(d)$ and $Cl_H(d')$ are disjoint subsets of B. Hence $\{\sum Cl_H(d) + I(\alpha) \mid d \in D,\ |Cl_H(d)| < \infty$ and $T_H(d) = \{1\}\}$ is K-linearly independent. So it is a basis for $C_{KG\alpha}(KH\alpha)$ as a K-module.

REMARK. From the proof Theorem 4, it is clear that the conclusions of the theorem remain valid if K is a commutative ring with 1 such that $1 - \alpha(n)$ is not a zero divisor in K for every $1 \neq n \in N$.

If $G = H$, then we obtain Corollary 5. Thus Theorem 4 generalizes Theorem 7 of [1].

COROLLARY 5. *Suppose K is an integral domain, G is a group, N is a central subgroup of G and $\alpha : N \to K^*$ is an injective homomorphism. Let $D \subseteq G$ and $\{N_d \mid d \in D\}$ be a transversal for the conjugacy classes of G/N. Then the set $\{\sum Cl(d) + I(\alpha) \mid d \in D, \ |Cl(d)| < \infty$ and $T(d) = [d, G] \cap N = \{1\}\}$ is a basis for the center of $KG\alpha$ as a K-module.*

References

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