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A ROLE OF CURVATURES IN THE CLASSIFICATION OF MANIFOLDS *

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0. Introduction

The category of isomorphism classes of differentiable manifolds of dimension ≤ 3 is equal to the category of isomorphism classes of topological manifolds of dimension ≤ 3, although for the higher dimensional case this is no longer true. For instance, there is a (unique) simply connected topological spin 4-manifold with the prescribed definite intersection form [F], but there is no such 4-manifold in the differentiable category [D]. This result is obtained by studying (nonlinear) Yang-Mills equation on an $SU(2)$ bundle. In the bundle theoretic point of view, the real line $\mathbf{R}^1$ and the circle S^1 are different, since the former has only the trivial bundles and the latter has the nontrivial (Möbius) line bundle. In the cohomological theoretic point of view, real line bundles over a topological manifold X corresponds to an element, called the first Stiefel-Whitney class, in $H^1(X;\mathbf{Z}_2)$ and the Möbius bundle corresponds to the generator of $H^1(S^1;\mathbf{Z}_2) \cong \mathbf{Z}_2$. The same type of reasoning is true for compact surfaces. For, if we denote by gX the connected sum of g copies of a surface X, then

$$H^1(gT^2;\mathbf{Z}_2) = (2g)\mathbf{Z}_2, \qquad H^1(g\mathbf{P}^2;\mathbf{Z}_2) = g\mathbf{Z}_2$$

where T^2 is the torus, $\mathbf{P}^2$ is the real projective plane and $g\mathbf{Z}_2$ is the direct sum of g copies of $\mathbf{Z}_2$.

In fact, the theory of vector bundles are closely related with the classification of 4-manifolds.

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THEOREM (M. FREEDMAN). *Two compact simply connected 4-manifolds are homeomorphic if and only if they have the isomorphic cohomology rings.*

The ring structure of the cohomology of such 4-manifold X is just the cup product

$$\cup : H^2(X;\mathbf{Z}) \otimes H^2(X;\mathbf{Z}) \to H^4(X;\mathbf{Z}),$$

where $H^2(X;\mathbf{Z})$ is equal to the isomorphism classes of complex line bundles on X and $H^4(X;\mathbf{Z})$ is equal to the isomorphism classes of quaternion line bundles on X. In this point of view, the above cup product is the following identity

$$c_1(L_1) \cup c_1(L_2) = c_2(L_1 \oplus L_2),$$

for complex line bundles L_1 and L_2, where c_i is the i-th Chern class.

For the high dimensional case, we still have the concept of anti-self-dual connections when the base manifold is a hermitian manifold. Anti-self-dual connections are special types of Einstein connections [Kob]. Although many statements in this paper are true for Einstein connections, we will focus only on anti-self-dual connections.

1. Anti-self-dual connections on high dimensional hermitian manifolds

Let M be a compact complex manifold of (complex) dimension $n \geq 2$ equipped with a hermitian metric

$$g = \sum_{\mu,\nu=1}^{n} g_{\mu\bar{\nu}} dz^{\mu} \otimes d\bar{z}^{\nu}. \tag{1.1}$$

The associated (real) fundamental 2-form is denoted by Φ;

$$\Phi = \sqrt{-1} \sum_{\mu,\nu=1}^{n} g_{\mu\bar{\nu}} dz^{\mu} \wedge d\bar{z}^{\nu}. \tag{1.2}$$

The space of real differential k-forms on M is denoted by A^k for $k = 0, 1, \cdots, 2n$. Then

$$A^k \otimes \mathbf{C} = \sum_{p+q=k} A^{p,q}, \tag{1.3}$$

where $A^{p,q}$ denotes the space of complex differential forms on M of type (p,q).

DEFINITION 1.4. A primitive real (1,1)-form on M is said to be *anti-self-dual.*

Thus $\omega = \sqrt{-1}\sum_{\mu,\nu=1}^{n} \omega_{\mu\bar{\nu}} dz^\mu \wedge d\bar{z}^\nu$ is anti-self-dual if and only if

$$\Lambda\omega := \sum_{\mu,\nu=1}^{n} g^{\mu\bar{\nu}}\omega_{\mu\bar{\nu}} = 0,$$

where $(g^{\mu\bar{\nu}})$ is the inverse matrix of $(g_{\mu\bar{\nu}})$; $\sum_{\nu=1}^{n} g^{\mu\bar{\nu}} g_{\sigma\bar{\nu}} = \delta^\mu_\sigma$. Thus the space A^2 of 2-forms on M decomposes into two orthogonal subspaces;

$$A^2 = A^2_+ \oplus A^2_-, \tag{1.5}$$

where

$$A^2_+ \otimes \mathbf{C} = A^{2,0} \oplus A^{0,2} \oplus \{f\Phi : f \in C^\infty(M) \otimes \mathbf{C}\}. \tag{1.6}$$

If $d\Phi^{n-2} = 0$ (in particular, when M is Kähler) , by Hodge theory, the above decomposition is true also in the cohomology level;

$$H^2(M;\mathbf{R}) = H^2_+ \oplus H^2_-, \tag{1.7}$$

where $H^2_\pm$ is the space of (anti-)self-dual closed 2-forms on the hermitian manifold (M, Φ).

Now let E be a smooth complex vector bundle over M of $\mathrm{rank}_{\mathbf{C}} = r$. We denote by $A^k(E)$ (resp. $A^{p,q}(E)$) the space of differential k-forms (resp. (p,q)-forms) on M with values in E. Then

$$A^k(E) = \sum_{p+q=k} A^{p,q}(E).$$

Now a connection

$$D : A^0(E) \to A^1(E) \tag{1.8}$$

is said to be ***anti-self-dual*** if the associated curvature tensor

$$R = R_+ + R_- \in A^2(\mathrm{End}(E)) = A^2_+(\mathrm{End}(E)) \oplus A^2_-(\mathrm{End}(E)) \tag{1.9}$$

is anti-self-dual , i.e., $R_+ = 0$. In particular, every flat bundle admits an anti-self-dual connection. Anti-self-dual connections are special types of Einstein-Hermitian connections [Kob]. If E admits an anti-self-dual connection and $d\Phi^{n-1} = 0$, then it is obvious that

$$c_1(E) \cup [\Phi^{n-1}] = 0 \in H^{2n}(M; \mathbf{R}).$$

Now the following proposition is trivial.

PROPOSITION 1.10. *If D_1 (resp. D_2) is an anti-self-dual connection on a vector bundle E_1 (resp. E_2) over a hermitian manifold M, then*

$$D_1 \oplus D_2, \quad D_1 \otimes D_2, \quad D_1^*$$

are anti-self-dual connections on $E_1 \oplus E_2$, $E_1 \otimes E_2$ and E_1^*, respectively.

Now let h be a hermitian metric on E and let D be an anti-self-dual connection on E compatible with h. Then the curvature R of D is of type $(1,1)$ and hence E *admits a unique holomorphic structure $\mathcal{E} = E^D$ such that D is the associated Chern connection* [AHS]. Recall that a holomorphic vector bundle $\mathcal{E}$ over a complex manifold M is said to be *simple* [OSS] if constant endomorphisms are the only holomorphic endomorphisms of $\mathcal{E}$, or equivalently,

$$H^0(M, sl(\mathcal{E})) = 0,$$

where $sl(\mathcal{E})$ is the bundle of trace-free endomorphisms of $\mathcal{E}$;

$$\mathrm{End}(\mathcal{E}) = sl(\mathcal{E}) \oplus \mathbf{C} \cdot 1_{\mathcal{E}}.$$

Now the following proposition is a special case of the more general vanishing principle [Kob, p.52].

PROPOSITION 1.11. *Let D be an anti-self-dual connection on a unitary vector bundle (E, h). Then any holomorphic section of $\mathcal{E} = E^D$ is parallel, and $\mathcal{E}$ is a direct sum of simple bundles. If D is irreducible, then $\mathcal{E}$ is simple.*

2. Determinant line bundle and the moduli space

Let (E, h) be a rank r hermitian vector bundle over M. Then the line bundle $\det(E) = \Lambda^r E$ is equipped with the induced metric $\det(h)$. The set of all connections on (E, h) will be denoted by $Con(E, h)$ and the subspace of $Con(E, h)$ consisting of anti-self-dual connections is denoted by $ASD(E, h)$. A connection (resp. anti-self-dual connection) D on (E, h) induces a connection (resp. anti-self-dual connection) $\det(D)$ on $(\det(E), \det(h))$ and hence we have the following commutative diagram

$$\begin{array}{ccc} ASD(E,h) & \longrightarrow & Con(E,h) \\ {\scriptstyle\det}\downarrow & & \downarrow{\scriptstyle\det} \\ ASD(\det(E),\det(h)) & \longrightarrow & Con(\det(E),\det(h)) \end{array}$$

with the surjective vertical arrows [K2]. Now we define for each $\nabla \in Con(\det(E), \det(h))$,

$$Con(E, h, \nabla) = \{D \in Con(E, h) : \det(D) = \nabla\}$$

and

$$ASD(E, h, \nabla) = ASD(E, h) \cap Con(E, h, \nabla).$$

Then

$$Con(E, h) = \cup\{Con(E, h, \nabla) : \nabla \in Con(\det(E), \det(h))\}$$

and

$$ASD(E, h) = \cup\{ASD(E, h, \nabla) : \nabla \in ASD(\det(E), \det(h))\}.$$

Now let $U(E,h)$ be the group of all C^∞ isometries on (E,h) and let $SU(E,h)$ be the subgroup of $U(E,h)$ consisting of elements with determinant $= 1$. Then $SU(E,h)$ acts on each $Con(E,h,\nabla)$ and the subspace $ASD(E,h,\nabla)$ is invariant. The quotient space

$$\mathcal{M} = \mathcal{M}(E,h,\nabla) = ASD(E,h,\nabla)/SU(E,h)$$

is called the *moduli space of anti-self-dual connections on E.*

PROPOSITION 2.1. *$\mathcal{M}$ is independent of the choice (h,∇).*

PROOF: Note that any two hermitian structures h and h' are equivalent, i.e., there exists a C^∞ bundle automorphism $f : E \to E$ such that $h' = f^*(h)$. Now this automorphism also pulls back connections;

$$f^* : Con(E,h) \to Con(E,h'),$$

given by $f^*(D) = f^{-1} \circ D \circ f$ for $D \in Con(E,h)$. Obviously

$$f^*(ASD(E,h)) \subset ASD(E,h').$$

Note that for $g \in U(E,h)$,

$$f^*(g) := f^{-1} \circ g \circ f \in U(E,h')$$

and the diagram

$$\begin{array}{ccc} Con(E,h) & \xrightarrow{f^*} & Con(E,h') \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f^*(g)} \\ Con(E,h) & \xrightarrow[f^*]{} & Con(E,h') \end{array}$$

commutes. Now for $\nabla \in Con(\det(E), \det(h))$, let $f^*(\nabla) = (\det\ f^{-1}) \circ \nabla \circ (\det\ f)$. Then

$$f^* : Con(E,h,\nabla)/SU(E,h) \simeq Con(E,f^*(h),f^*(\nabla))/SU(E,f^*(h)).$$

If $\nabla \in ASD(\det(E), \det(h))$, then $f^*(\nabla) \in ASD(\det(E), \det(f^*(h))$ and

$$f^* : \mathcal{M}(E, h, \nabla) \simeq \mathcal{M}(E, f^*(h), f^*(\nabla)).$$

Now to complete the proof, we consider two connections $\nabla_1, \nabla_2 \in Con(\det(E), \det(h))$. Then $\nabla_2 = \nabla_1 + \sqrt{-1}\phi$ for some real 1-form ϕ on M. Now the translation

$$+\sqrt{-1}(\phi/r) \cdot 1_E : Con(E, h, \nabla_1) \to Con(E, h, \nabla_2)$$

commutes with the action of the gauge group $SU(E, h)$ and hence we obtain the identification

$$Con(E, h, \nabla_1)/SU(E, h) \simeq Con(E, h, \nabla_2)/SU(E, h).$$

If $\nabla_1, \nabla_2 \in ASD(\det(E), \det(h))$, then we obtain the identification

$$\mathcal{M}(E, h, \nabla_1) \simeq \mathcal{M}(E, h, \nabla_2).$$

This completes the proof. ∎

3. Elliptic complex

The space $Con(E, h, \nabla)$ of connections is a 'parallel translation' of $A^1(su(E, h))$, where $su(E, h)$ is the *real* vector bundle of trace free skew-hermitian endomorphisms of (E, h) so that

$$su(E, h) \otimes \mathbf{C} = sl(E),$$

and $A^0(su(E, h))$ is the Lie algebra of the gauge group $SU(E, h)$. For any connection D on E, the induced connection on $su(E, h)$ is still denoted by D;

$$D : A^0(su(E, h)) \to A^1(su(E, h)).$$

Now if D is anti-self-dual, i.e., $D \in ASD(E, h, \nabla)$, then

$$D + \alpha, \qquad \alpha \in A^1(su(E, h))$$

is anti-self-dual if and only if

$$p_+(D(\alpha) + \frac{1}{2}[\alpha, \alpha]) = 0, \tag{3.1}$$

where $p_+ : A^2 \to A^2_+$ is the projection. The linearization of the above equation gives rise to a map

$$D_+ = p_+ \circ D : A^1(su(E,h)) \to A^2_+(su(E,h)). \tag{3.2}$$

Now we define

$$D_2 : A^2_+(su(E,h)) \to A^{0,3}(sl(E)) \tag{3.3}$$

as the following composition

$$\begin{aligned} A^2_+(su(E,h)) \hookrightarrow A^2(su(E,h)) \hookrightarrow A^2(sl(E)) \\ \xrightarrow[p_{0,2}]{} A^{0,2}(sl(E)) \xrightarrow[D'']{} A^{0,3}(sl(E)), \end{aligned}$$

where $p_{0,2}$ is the projection and D'' is the $(0,1)$-part of the connection D;

$$D = D' + D''.$$

Now we have

THEOREM 3.4 [K1]. *Let D be an anti-self-dual connection on a hermitian vector bundle (E, h) over a hermitian manifold M. Then the sequence*

$$\begin{aligned} 0 \to A^0(su(E,h)) \xrightarrow{D} A^1(su(E,h)) \xrightarrow{D_+} A^2_+(su(E,h)) \xrightarrow{D_2} A^{0,3}(sl(E)) \\ \xrightarrow[D'']{} A^{0,4}(sl(E)) \xrightarrow[D'']{} \cdots \xrightarrow[D'']{} A^{0,n}(sl(E)) \to 0 \end{aligned}$$

is an elliptic complex with the index equal to

$$2\, ch(sl(E)) \cup todd(M)$$

evaluated on the fundamental cycle of the manifold M, where $todd(M)$ denotes the Todd class of M.

Moreover, its cohomology groups H^* satisfy

$$\begin{array}{ll} H^0 \otimes \mathbf{C} \simeq H^0(M, sl(\mathcal{E})) & H^1 \simeq H^1(M, sl(\mathcal{E})) \\ H^2 \simeq H^0 \oplus H^2(M, sl(\mathcal{E})) & H^k \simeq H^k(M, sl(\mathcal{E})) \quad \textit{for } k \geq 3, \end{array}$$

where $\mathcal{E} = E^D$ is the holomorphic structure associated with D.

The proof may be also found in [Kob, p.248].

4. Anti-self-dual connections on a complex homology 6-sphere

It is known that S^6 is Yang-Mills instable [KOT]. On the other hand, anti-self-dual connections on a Kähler manifold are absolute minima of the Yang-Mills functional.

Now let (M, Φ) be a hermitian manifold of (complex) dimension 3 with the trivial canonical line bundle $\mathcal{K}_M$ such that

$$H^*(M; \mathbf{Z}) \simeq H^*(S^6; \mathbf{Z}).$$

Let (E, h) be a smooth hermitian vector bundle of rank 3 with $c_3(E) \neq 0$. Note that if $E = E_1 \oplus E_2$ with $\mathrm{rank}(E_i) = i$, then

$$c_3(E) = c_1(E_1) \cup c_2(E_2)$$

and hence $c_3(E) \neq 0$ insures that E is smoothly indecomposable. Therefore, every connection on (E, h) is irreducible and hence any holomorphic structure on E is simple, i.e., $H^0(M, sl(\mathcal{E})) = 0$ and, by Serre duality, $H^1(M, sl(\mathcal{E})) \simeq H^2(M, sl(\mathcal{E}))$ and $H^3(M, sl(\mathcal{E})) = 0$.

Therefore, if D is an anti-self-dual connection, the cohomology groups of the elliptic complex (3.4) satisfy

$$H^0 = 0, \quad H^1 \simeq H^2, \quad H^3 = 0$$

and the index of the elliptic complex is zero. Note that $H^1(M, sl(\mathcal{E}))$ is the tangent space of the moduli space when $H^2(M, sl(\mathcal{E})) = 0$ [K1] and hence in this case the moduli space is discrete.

REFERENCES

[AHS] M. F. Atiyah, N. J. Hitchin and I. M. Singer, *Self-duality in four dimensional Riemannian geometry*, Proc. Roy. Soc. London A **362** (1978), 425–461.

[D] S. K. Donaldson, *An application of gauge theory to four dimensional topology*, J. Differential Geometry **18** (1983), 279–315.

[F] M. H. Freedman, *The topology of four dimensional manifolds*, J. of Differential Geometry **17** (1982), 357–453.

[K1] H.-J. Kim, *Moduli of Hermite-Einstein Vector Bundles*, Math. Z. **195** (1987), 143–150.

[K2] ———, *Reduced gauge groups and the moduli space of stable bundles*, J. Korean Math. Soc. **25** (1988), 259–264.

[Kob] S. Kobayashi, "Differential Geometry of Complex Vector Bundles," Publ. Math. Soc. Japan 15, Iwanami Shoten and Princeton Univ. Press, Tokyo, 1987.

[KOT] S. Kobayashi, Y. Ohnita and M. Takeuchi, *On Instability of Yang-Mills Connections*, Math. Z. **193** (1986), 165–189.

[OSS] C. Okonek, M. Schneider and H. Spindler, "Vector Bundles on Complex Projective Spaces," Progress in Math. 3, Birkhäuser, Boston, Basel, Stuttgart, 1980.

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