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# A STUDY ON THE EVALUATION OF NUMERICAL VALUES IN AN ALGEBRAIC EQUATION

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## 1. Reformed new–muller's method

Let $F(X)$ be a continuous function which has a root in an interval $[X_0, X_1]$.

First select two initial approximations $X_0,\ X_1$ such that $F(X_0)F(X_1) < 0$. Take $X_2 = (X_1 + X_1)/2$.

For the simplification, we put $F(X_n) = Y_n$ ($n = 0, 1, 2, 3, \cdots$) and $h = X_2 - X_0 = X_1 - X_2 = (X_1 - X_0)/2$. Take the quadratic polynomial

$$G(X) = a(X - X_2)^2 + b(X - X_2) + c \tag{1}$$

which interpolates the original $F$ at three point $X_0,\ X_1$ and $X_2$. The coefficients of $G$ are given by

$$a = \frac{Y_0 + Y_1 - 2Y_2}{2h^2}, \quad b = \frac{Y_1 - Y_0}{2h}, \quad c = Y_2 \tag{2}$$

We assume that $F(X)$ is analytic. Then we have

$$\begin{aligned} Y_0 &= Y_2 - hY_2' + \frac{h^2}{2}Y_2'' - \frac{h^3}{6}Y_2''' + \frac{h^4}{24}Y_2^{(4)} - \cdots \\ Y_1 &= Y_2 + hY_2' + \frac{h^2}{2}Y_2'' + \frac{h^3}{6}Y_2''' + \frac{h^4}{24}Y_2^{(4)} + \cdots \end{aligned} \tag{3}$$

where $Y_2',\ Y_2'', \cdots$ are the successive derivatives of $F$ at $X_2$. We have easily that

$$a = \frac{1}{2}Y_2'' + \frac{h^2}{24}Y_2^{(4)} + \cdots, \quad b = Y_2' + \frac{h^2}{6}Y_2''' + \cdots, \quad c = Y_2. \tag{4}$$

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Now, let $X_3$ be the root of $G(X) = 0$ close to $X_2$, say

$$(5) \qquad X_3 = X_2 - \frac{2c}{b + sgn(b)\sqrt{b^2 - 4ac}}.$$

We adopt this to avoid the cancelleration.

We select the next interval $I = [X_0', X_1']$ under condition $F(X_0')F(X_1') < 0$, as follows :

$1^o$ If $F(X_2)F(X_1) < 0$
$F(X_2)F(X_3) < 0 \Longrightarrow X_0' = X_2',\ X_1' = X_3,\ i.e., I = [X_2, X_3]$,
otherwise, $I = [X_3, X_1]$.

$2^o$ If $F(X_0)F(X_2) < 0$
$F(X_2)F(X_3) < 0 \Longrightarrow X_0' = X_3,\ X_1' = X_2,\ i.e., I = [X_3, X_2]$,
otherwise, $I = [X_0, X_3]$.

In every case, the next closed interval $I = [X_0', X_1']$ is chosen by the ordered pair $X_i$, which gives different signs of $F(X_i)$, $(i = 1, 2, 3$, or $i = 0, 2, 3)$.

This selection asserts, that in each case, the length of $I = [X_0, X_1]$ is less than the half of the one to the original interval $[X_0, X_1]$.

Hence this method asserts the convergence of the interval containing the zero point of (1).

After a few steps which reduce the length of interval enough, we assume that the derivatives of $F(X)$ are nearly constant.

Since we have selected $X_3$ as the zero point of (1), we obtain easily the following theorem.

THEOREM 1.1. *Let $F(X)$ be analytic in the interval $[X_0, X_1]$, which contains a simple zero $\alpha; F(\alpha) = 0$. Also we introduce the quadratic function $G(X)$ as in (2), and let $G(X_3) = 0$. Then $X_3$ converges to $\alpha$ cubically.*

*Proof.* We have by the definition

$$F(X_3) = Y_2 + thY_2' + \frac{1}{2}t^2h^2Y_2'' + \frac{1}{6}t^3h^3Y_2''' + \frac{1}{24}t^4h^4Y_2^{(4)} + \cdot$$

$$G(X_3) = Y_2 + thY_2' + \frac{1}{2}t^2h^2Y_2'' + \frac{1}{6}th^3Y_2''' + \frac{1}{24}t^2h^4Y_2^{(4)} + \cdots = 0.$$

Where $t = (X_3 - X_2)/h$, $|t| < 1$. From the above expressions, we find

$$F(X_3) = \frac{1}{6}h^3Y_2'''t(t^2-1) + \frac{1}{24}h^4Y_2^{(4)}t^2(t^2-1) + \cdots .$$

Since we have assumed that $F'(\alpha) \neq 0$, $F(X_3) = 0(h^3)$ implies $|X_3 - \alpha| = 0(h^3)$, i.e., the cubic convergence.

In 2, we discuss another method in calculating the degree of convergence.

## 2. The convergence of the new–muller's method

We use the notations as in the previous section. It is easy to see that $|h| \to 0$ for $n \to \infty$ and

$$a \doteqdot \frac{1}{2}Y_2'', \quad b \doteqdot Y_2', \quad c = Y_2$$

from (4). We have the following (7) from (5) and (6), provided that $Y_2Y_2''$ is much smaller than $(Y_2')^2$.

$$X \doteqdot X - \frac{2Y_2Y_2'}{2(Y_2')^2 - Y_2Y_2''} \tag{7}$$

To examine the degree of convergence for (7), let $\alpha$ be a root of $F$ and put

$$\psi(X) = X - \frac{2YY'}{2(Y')^2 - YY''} \tag{8}$$

where $Y = F(X)$, and $Y'$, $Y''$ are it's derivatives.

Then, the procedure to take $X_3$ as the next approximation may be written in the form of $X_{n+1} = \psi(X_n)$, at the final stages so that

$$\begin{aligned} X_{n+1} - \alpha &= \psi(X_n) - \psi(\alpha) \\ &= \psi'(\alpha)(X_n - \alpha) + \frac{1}{2}\psi''(\alpha)(X_n - \alpha)^2 + \frac{1}{6}\psi'''(\alpha)(X_n - \alpha)^3 + \cdots \\ &= \frac{1}{6}\psi'''(\alpha)(X_n - \alpha)^3 + \cdots \end{aligned}$$

because, we have $\psi'(\alpha) = \psi''(\alpha) = 0$. We see

$$\psi'''(\alpha) = \{Y'''(\alpha) - 1/2Y''(\alpha)^2\}/Y'(\alpha)^2.$$

This implies that the degree of convergence of the iteration by (8) is cubic, contrasting that the degree of convergence for the Muller's Method is degree of 1.84 (See [1]).

## 3. Examples of calculation

EXAMPLES.

1) $X^3 - X - 1 = 0$
2) $X^4 - 3X^3 - X^2 + 2X + 3 = 0$
3) $X^5 - 2X^4 - 4X^3 + X^2 + 5X + 3 = 0$
4) $X^6 - 8X^4 - 4X^3 + 7X^2 + 13X + 6 = 0$
5) $X^7 + X^6 - 8X^5 - 12X^4 + 3X^3 + 20X^2 + 19X + 6 = 0$

The results of the Muller's and New-Muller's Method are shown in the Table 1.

| Example No. | Degree of Equation | Accuracy of Last Root to be Found | | Times of Repeating | |
|---|---|---|---|---|---|
| | | Muller | New-Muller | Muller | New-Muller |
| 1) | 3 | $10^{-12}$ | $10^{-12}$ | 6 | 4 |
| 2) | 4 | $10^{-12}$ | $10^{-12}$ | 5 | 5 |
| 3) | 5 | $10^{-12}$ | $10^{-12}$ | 6 | 5 |
| 4) | 6 | $10^{-12}$ | $10^{-12}$ | 6 | 4 |

(Table 1)

In the computation of Example 5, we fail to have the solution in the Muller's Method as shown below in the Table 2.

| Solved case with needed root | Failed case |
|---|---|
| RUN-Number = 20 | RUN-Number = 20 |
| s(1)= 1.00000000 | s(1)= 1.00000000 |
| s(2)= 1.00000000 | s(2)= 1.00000000 |
| s(3)= -8.00000000 | s(3)= -8.00000000 |
| s(4)=-12.00000000 | s(4)=-12.00000000 |

| | |
|---|---|
| s(5)= 3.00000000 | s(5)= 3.00000000 |
| s(6)= 20.00000000 | s(6)= 20.00000000 |
| s(7)= 19.00000000 | s(7)= 19.00000000 |
| s(8)= 6.00000000 | s(8)= 6.00000000 |
| $X_0$=1.500000000000 | $X_0$=0.00000, $X_1$=0.50000 |
| $X_2$=2.000000000000 | $X_2$=1.00000 |
| $X_1$=2.500000000000 | XANS=-0.181953492716743103763121781 |
| XANS=1.486557539197503663430666165 | XANS=-0.297688107790282336928555651 |
| XANS=1.480369343312102958787335183 | XANS=-0.595205607486539223227595130 |
| XANS=1.475050097625060563366616861 | XANS=-0.812730423617922159706949969 |
| XANS=1.474989038025216081528867562 | XANS=-0.680250828874700885773307846 |
| XANS=1.474989038334796698226369926 | XANS=-0.686026232904809407653345943 |
| XANS=1.474989038334796698226369926 | XANS=-0.686002934602659308893635171 |
| 1.4749890383348 | XANS=-0.686002948238860277285766642 |
| | XANS=-0.686002948238860088547852456 |
| | -0.6860029482388 |
| F(x)=-1.7763568394003D-15 | F(x)=-1.1102230246252D-16 |

(Table 2)

## 4. Another solution using the circles of curvature

We give an alternative numerical method associated with the solution of an equation using the circles of curvature in $R^1$. The degree of convergence in this way is also cubic.

In some cases this method is effective, although Newton–Raphson's method dose not work well. For an equation in $R^1$

(9) $$F(X) = 0,$$

which is assumed $F \in C^2$, let $X$ be an approximate solution of (9).

As is seen, the curvature circles of $Y = F(X)$ at the point $(X_0, Y_0) = (X_0, F(X_0))$ is given such as

(10)
$$[X - X_0 + \frac{Y_0'\{(1+(Y_0')^2\}}{Y_0''}]^2 + \{Y - Y_0 - \frac{1+(Y_0')^2}{Y_0''}\}^2 = \frac{\{(1+(Y_0')^2\}^3}{(Y_0'')^2}$$

We take the coordinate $X'$ at which the curve given by the equation (10) crosses through the $X$–axis, for the next approximate solution.

In this way, we get an iterated formula evaluating the solution of (9) as follows

(11) $$X_{n+1} = X_n - \{Y_n^2 Y_n'' + 2Y_n(Y_n')^2\}\{(2Y_n')^3\}^{-1}$$

or

$$(12) \qquad X_{n+1} = X_n - C_n/\{B_n + sgn(B_n)(B_n - C_n)^{1/2}\}$$

Here,

$$\begin{aligned} Y_n &= F(X_n), \\ Y_n^{(i)} &= F^{(i)}(X_n), \\ B_n &= Y_n\{1 + (Y_n')^2\}/Y_n'', \\ C_n &= [Y_n^2 Y_n'' + 2Y_n\{1 + (Y_n')^2\}]/Y_n''. \end{aligned}$$

We call the equation (11) as the first formula and the equation (12) as the second formula.

THEOREM 4.1. *Let $I = [a, b]$ and let $F : I \to R^1$ be a five times differentiable mapping in the interval $I$. Suppose that $F(a)F(b) < 0$ and there are constants $m$ and $M$ such that $|F'(X)| \geq m > 0$ and $|F^{(n)}(X)| \leq M$ $(n = 0, 1, 2, 3, 4, 5)$ for all $X \in I$.*

*We put*

$$K = \frac{1}{6}\{\frac{M}{m} + \frac{M^2}{m^2} + \frac{33M^3}{2m^3} + \frac{21M^4}{2m^4} + \frac{36M^5}{m^5} + \frac{30M^6}{m^6}\}.$$

*Then there exists a subinterval $I^*$ containing a zero of the function $F$ such that for any $X_1 \in I^*$, the sequence $\{X_n\}$ defined by*

$$(*) \qquad X_{n+1} = X_n - \frac{F(X_n)^2 F''(X_n) + 2F(X_n)F'(X_n)^2}{2F'(X_n)^3}$$

*is contained in the interval $I$, and $\{X_n\}$ converges to $\alpha$. Moreover, we have*

$$(**) \qquad |X_{n+1} - \alpha| \leq K^2 |X_n - \alpha|^3 \quad for \quad n \in N.$$

*Which asserts the cubic convergence.*

*Proof.* Let $Y = F(X_n)$, $Y^{(i)} = F^{(i)}(X_n)$.
Put $\psi(X) = X - \{Y^2 Y'' + 2Y(Y')^2\}/\{2(Y')^3\}$.

Then the procedure (11) may be written in the form of

$$X_{n+1} = \psi(X_n),$$

so that

$$X_{n+1} - \alpha = \psi(X_n) - \psi(\alpha). \tag{13}$$

By Taylor's theorem there exists a point $C$ between $X_n$ and $\alpha$ such that

$$\psi(X_n) = \psi(\alpha) + \psi'(\alpha)(X_n - \alpha) + \frac{1}{2}\psi''(\alpha)(X_n - \alpha)^2 + \frac{1}{6}\psi'''(C)(X_n - \alpha)^3. \tag{14}$$

By the expression (13) and (14).

$$X_{n+1} - \alpha = \psi(\alpha)'(X_n - \alpha) + 1/2\psi''(\alpha)(X_n - \alpha)^2 + 1/6\psi'''(C)(X_n - \alpha)^3.$$

By direct computation, we have

$$\psi'(\alpha) = 0, \quad \psi''(\alpha) = 0,$$

and $|\psi'''(C)| \leq 6K^2$, so that we obtain the inequality

$$|X_{n+1} - \alpha| \leq K^2|X_n - \alpha|^3. \tag{15}$$

We now choose $\delta > 0$, so small that $\delta < 1/K$ and the interval $I^*[\alpha - \delta, \alpha + \delta]$ belongs to the original interval $I$.

If $X_n \in I^*$, then $|X_n - \alpha| \leq \delta$, and it follows from the expression (15), we see $|X_{n+1} - \alpha| \leq K^2|X_n - \alpha| \leq K^2\delta^3 < \delta$, hence $X_n \in I^*$ implies that $X_{n+1} \in I^*$.

Therefore if $X_1 \in I^*$, we infer that $X_n \in I^*$ for all $n \in N$. Also, if $X_1 \in I^*$, then $|X_{n+1} - \alpha| \leq (K\delta)^{3^n - 1}|X_1 - \alpha|$ for $n \in N$. But since $K\delta < 1$, this proves the convergence $X_n \to \alpha$. (Similarly, we can show the results for the second formula (12).)

REMARK 4.2. *In order the expression (11) to converge, the interval $I$ is required to satisfy $|Y'| > |Y|$, $|Y'| > |1/2Y''|$. Further, the interval $I$ should be the set which includes a zero point $\alpha$; $F(\alpha) = 0$.*

REMARK 4.3. *If at the first step, the circle of curvature dose not cross the $X$-axis, we take the extreme (Minimum or Maximum) point $X'$ as the next approximation. After few steps, the circle of curvature dose cross the $X$-axis, provided that there is a zero point $\alpha$ in the interval $I$.*

## 5. Numerical examples

EXAMPLE 1. *The Table 3 shows the results of computation by Newton –Raphson's Method and the second method in 4 applied to the equation $SIN(2.1X - 0.6) = 0$ with the initial approximation $X_0 = 1$. Even when Newton–Raphson's Method doesn't work, it may be possible to have a solution by this method.*

The same situation occurs for the value $X_0$ in an interval [1,2].

| A Number of iteration | Newton–Raphson | Curvature Iteration |
|---|---|---|
| 1 | −5.7149519920349 | 1.69681475779063 |
| 2 | −5.6982652358711 | 1.78123734436566 |
| 3 | −5.6982718813119 | 1.78171084762747 |
| 4 | −5.6982721689230 | |
| | ...... | (Second Formula) |
| F(x) | Failure | −8.7422783679D-08 |

(Table 3)

*Here, 1.0 is given as an initial approximation.

COMMENT: This phenomena is reasonable, since all zero points are the points of reflection.

EXAMPLE 2. *In Table 4, we compare the method (11) and (12) with the Newton Raphson's Method for the following equations.*

1) $F(X) = X^3 - X^2 + 1 = 0$
2) $F(X) = X^4 - X^3 - X^2 + 2X + 3 = 0$

3) $F(X) = X^5 - 2X^4 - 4X^3 + X^2 + 5X + 3 = 0$
4) $F(X) = X^6 - 8X^4 - 4X^3 + 7X^2 + 13X + 6 = 0$
5) $F(X) = X^7 + X^6 - 8X^5 - 12X^4 + 3X^3 + 20X^2 + 19X + 6 = 0$

| Example of No. | Degree Equation | Accuracy of the last Root to be Found | | | Number of Iteration | | |
|---|---|---|---|---|---|---|---|
| | | Newton | First Formula | Second Formula | Newton | First Formula | Second Formula |
| 1) | (*1) 3 | E-16 | E-16 | E-16 | 5 | 4 | 4 |
| 2) | (*1) 4 | E-16 | E-16 | E-16 | 5 | 3 | 3 |
| 3) | (*1) 5 | E-14 | E-15 | E-15 | 4 | 4 | 4 |
| 4) | (*1.5) 6 | E-15 | E-15 | E-15 | 3 | 2 | 2 |
| 5) | (*1.5) 7 | E-15 | E-14 | E-15 | 3 | 2 | 2 |

(* Initial Approximations) (Table 4)

As is confirmed in the Table 4, the number of iteration in our method does not exceed to the one for the Newton–Raphson's Method.

## 6. An initially approximation for the equation of the sixth degree

The initial approximation plays an important role for the successive iteration method. Actually, the convergence itself critically depends upon the value of initial approximation. Therefore, it is very important to study the problem of choosing a good initial approximation.

We propose an initial approximation for the equation (16) of the sixth degree without the fifth term;

$$X^6 + b_4X^4 + b_3X^3 + b_2X^2 + b_1X + b_0 = 0 \tag{16}$$

General equation of the sixth degree can be transformed in this type by translation. We assume that (16) is factorized as following (17);

$$(X^3 + \alpha_1X + \beta_1)(X^3 + \alpha_2X + \beta_2). \tag{17}$$

Then, the coefficients $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$ are obtained from the following quadratic equation;

$$\alpha^2 - b_4\alpha + b_2 = 0, \quad \beta^2 - b_3\beta + b_0 = 0.$$

When $\alpha_1\beta_2 + \alpha_2\beta_1 = b_1$, the roots of the equation (17) = 0 coincide with the roots of the original equation (16). Thus, we may adopt as the initial approximations the solution of two cubic equations

$$X^3 + \alpha_1 X + \beta_1 = 0, \quad X^3 + \alpha_2 X + \beta_2 = 0.$$

The nearer the value of $\alpha_1\beta_2 + \alpha_2\beta_1$ is close to $b_1$, the more they are suitable as the initial approximation for the solution of equation (16).

## 7. Numerical examples

EXAMPLE 1. *To solve a practical problem, we take an equation of the sixth degree*

$$X^6 - 28X^4 - 14X^3 - 147X^2 - 14X - 120 = 0 \tag{18}$$

*According to the procedure in section 6, we compute the equation (17), and we get* $\alpha_1 = -7$, $\alpha_2 = 21$, $\beta_1 = -6$, $\beta_2 = 20$. *In this case the condition* $\alpha_1\beta_2 + \alpha_2\beta_1 = b_1$ *is satisfied. We solve the equations*

$$X^3 - 7X - 6 = 0, \quad X^3 - 21X + 20 = 0 \tag{19}$$

*by means of the Cardano's Formula, which gives*

$$X_1 = 3,\ X_2 = -1,\ X_3 = -2,\ X_4 = -5,\ X_5 = 4,\ X_6 = 1.$$

*Since the equation (18) satisfies the condition* $\alpha_1\beta_2 + \alpha_2\beta_1 = b_1$, *the above solution are exact with no errors for the equation (18).*

EXAMPLE 2. *As the special case of* $n = 3$ *of the Equation concerning spin–glass, we have the following equation (20) after the other variables are eliminated. (See the References [6].)*

(20)

$$77X^6 - 688X^5 + 2509X^4 - (42872/9)X^3 + 4935X^2 - 2616X + 543 = 0,$$

*This equation (20) has four real–roots and pair of complex roots at the equation (20). If we put* $X = \{2/(T+2)\} + 1$, *we get a simplified equation;*

$$T^6 - 30T^4 + 72T^3 - 96T^2 + 18T + 26 = 0. \tag{21}$$

*We take the initial values for the equation (21) as in the Section 6. (See the Table 5). The practical calculation of the equation (21) by means of the method described in Section 6 are as follows ;*

$$(22)\qquad \begin{aligned} \alpha_1 &= 15 - \sqrt{321} = -32.91647287, \\ \alpha_2 &= -15 + \sqrt{321} = 2.916472887, \\ \beta_1 &= 36 + \sqrt{1270} = 71.63705936, \\ \beta_2 &= 36 - \sqrt{1270} = 0.362940638. \end{aligned}$$

*We can rewrite the above expression (22) as follows ;*

$$(23)\qquad T^3 - 32.91647287\,T + 71.63705936 = 0$$

$$(24)\qquad T^3 + 2.916472887\,T + 0.362940638 = 0$$

*The Table 5 shows the calculation of the equations (23) and (24).*

| Solution of the Equation (23) and (24) | Solution by New Muller's Method | Number of Iteration | Radical of X for (20) | |
|---|---|---|---|---|
| −6.6141750 | −6.57435579740 | 3 | 0.562779965403 | |
| 3.6326749 | 4.24956154493 | 4 | 1.32002245048 | (25) |
| 2.9815271 | 0.864403949676 | 4 | 1.69822554191 | |
| 0.1237946 | −0.385578818797 | 3 | 2.23883409317 | |
| 0.0618973 | 0.922984560779 | * | 1.55760144206 | (26) |
| ±1.7111301 i | ±1.39294538426 i | | ±0.265724412436 i | |

(Table 5)

REMARK 7.1. *The solution (25) and (26) are the roots of the equation (23) and (24) respectively. Also, the error bound for the convergence is* $\epsilon = 10^{-7}$.

The Table 6 shows the solutions by the Durand–Kerner's Method, starting from the same values in the left–most column of Table 5.

| Times of Iteration | Solution by Durand–Kerner's Method<br>Solution of the Equation (21) |
|---|---|
| 6 | 0.922984560779 ± 1.39294538426 i |
| 1 | −6.57435579737 |
| 2 | −0.385578818797 |
| 3 | 0.864403949674 |
| 4 | 4.24956154493 |

(Table 6)

**Note** It is shown that the first line is the complex roots, and the other lines are the real roots.

## 8. Conclusion

I. The New–Muller's Method is different from the conventional Muller's Method in the following point:

i) The New–Muller's Method begins with two initial approximations $X_0$, $X_1$ and their mean value $X_2$ is used as intermediate initial approximation.

ii) As is shown in the Table 2, the New–Muller's Method has wider intervals to be the initial values than the one of the Muller's Method.(It is seen by the intermediate value theorem).

iii) The New–Muller's Method is simpler than the conventional Muller's Method in the calculation formula. Further the degree of convergence for New–Muller's Method is cubic while Muller's Method is degree of 1.84.

II. The formula for iteration method using the circle of curvature is relatively simpler and it's convergence is faster than the case of Newton's method.

III. As for the Equation concerning spin–glass which is complicated in the solution, it can be solved by using the initial value proposed in this paper.

As a conclusion, it is proposed A NEW FORMULA of which it's convergence rate is faster than others. By applying the proposed NEW FORMULA, we can estimate simply the initial approximation in such case of the Equation concerning spin–glass.

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