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BOUNDARY VALUE PROBLEM FOR NON-LINEAR DISSIPATIVE HYPERBOLIC EQUATIONS WITH SUPERLINEAR GROWTH NONLINEARITY *

WAN SE KIM AND OUI YOON WOO

1. Introduction

Let Z^+, Z and R be the set of all positive integers, integers and real numbers, respectively, and let $\Omega = [0, 2\pi] \times [0, \pi]$ and $I = [0, \pi]$.

Let $p \in [1, \infty[$. By $L^p(\Omega)$ we denote the space of all measurable real valued functions $u : \Omega \to R$ for which $|u(t,x)|^p$ is Lebesgue integrable. The norm is given by

$$\|u\|_{L^p} = [\iint_\Omega (t,x)|^p \, dt\, dx]^{\frac{1}{p}}.$$

In particular, $L^2(\Omega)$ is a space having usual inner product (,) and usual norm $\|\cdot\|_{L^2}$. Let $L^\infty(\Omega)$ be the space of measurable real valued functions $u : \Omega \to R$ which are essentially bounded with the norm

$$\|u\|_{L^\infty} = \operatorname*{ess\,sup}_{(t,x)\in\Omega} |u(t,x)| \, .$$

Let $C^k(\Omega)$ be the space of all continuous functions $u : \Omega \to R$ such that the partial derivatives up to order k with respect to both variables are continuous on Ω, while $C(\Omega)$ is used for $C^0(\Omega)$ with the usual norm $\| \ \|_\infty$ and we write $C^\infty(\Omega) = \cap_{k=0}^{\infty} C^k(\Omega)$.

Let $W^{k,2}(\Omega)$ be the Sobolev space of all functions $u : \Omega \to R$ in $L^2(\Omega)$ such that all distributional derivatives $D_t^p D_x^q$ $(0 \le p + q \le k)$ belongs to $L^2(\Omega)$ and the norm is given by

$$\|u\|_{W^{k,2}} = [\sum_{0 \le p+q \le k} \iint_\Omega [D_t^p D_x^q u(t,x)]^2 \, dt\, dx]^{\frac{1}{2}}.$$

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In this note, we will investigate the existence of weak solution of the periodic–Dirichlet problem for non–linear dissipative hyperbolic equations of the form

$$(1.1) \qquad \beta u_t + u_{tt} - u_{xx} + g(x,u) = h(t,x) \text{ in } \Omega$$

where $\beta(\neq 0) \in R$, $u = u(t,x)$, $h \in L^2(\Omega)$ and $g : I \times R \to R$ is a Caratneodory function, that is, $g(\cdot, u)$ is measurable on I for each $u \in R$ and $g(x,\cdot)$ is continuous on R with the continuity uniform with respect to a.e. $x \in I$. This holds, for example, if g is continuous on $I \times R$, but it also holds in many other case ; e.g., if $g(x,u) = p(x)F(u)$ where $p \in L^\infty(I)$ and F is continuous. Moreover, we assume that for each $d > 0$ there exists a constant $M_d > 0$ such that $|g(x,u)| \leq M_d$ for all $(x,u) \in I \times R$ with $|u| \leq d$.

A *weak solution* of the periodic–Dirichlet problem on Ω for (1.1) will be a $u \in L^\infty(\Omega)$ such that

$$(1.2) \qquad \iint_\Omega u(t,x)\big[-\beta v_t(t,x) + v_{tt}(t,x) - v_{xx}(t,x)\big]\,dt\,dx + \iint_\Omega g\big(x,u(t,x)\big)v(t,x)dt\,dx = \iint_\Omega h(t,x)v(t,x)dt\,dx$$

for every $v \in C^2(\Omega)$ satisfying boundary conditions

$$v(t,0) = v(t,\pi) = 0,\ t \in [0,2\pi]$$

$$v(0,x) - v(2\pi,x) = v_t(0,x) - v_t(2\pi,x) = 0,\ x \in [0,\pi].$$

Here we remark that a necessary condition for (1.2) to have a meaning is that g be such that $g(\cdot, u(\cdot,\cdot)) \in L^2(\Omega)$ when $u \in L^2(\Omega)$.

Our results lie in that we allow $g(x,u)$ to grow superlinearly in u when g satisfies a sign condition. The rate of growth allowed in g is any polynomial growth ; i.e., there exist $a(x)$, $b(x)$ in $L^\infty(\Omega)$

$$(H_1) \qquad |g(x,u)| \leq a|u|^p + b \text{ for } x \in I \text{ and } |u| \geq d_0 \text{ and for } p > 0.$$

We don't need any restriction on h except $h \in L^2(\Omega)$.

As an example, we have the following as a corollary of our main result. The periodic–Dirichlet boundary value problem

$$\beta u_t + u_{tt} - u_{xx} + F(x)sgn(u)|u|^p = h(t,x), \ F \in L^\infty(I), \ \beta \neq 0, \ p > 0,$$

and $u = u(t,x)$, has a weak solution for each $h \in L^2(\Omega)$.

Our condition used here is a kind of sign conditions ; i.e., there exists a function $\psi : R \to R$ such that $\limsup_{|u|\to\infty} \psi(|u|)/|u| = \alpha_0$, $(-\infty < \alpha_0 < \infty)$ with $ug(x,u) \geq -\psi(|u|)$ for all $(x,u) \in I \times R$. This condition is a weakening of the condition that $ug(u) \geq 0$ for large $|u|$ and is a strong tool in our proof of the main theorem. The latter condition is used by several authors. For example, in [3], Brezis and Nirenberg prove the existence of solutions for the periodic–Dirichlet problem for non-linear dissipative hyperbolic equations with sublinear growth in g and a Landesman–Lazer type condition and the same sign condition which we will impose. In [12], Mckenna and Rauch study elliptic boundary problems with a sign condition and a Landesman–Lazer type condition and without any restriction of the growth in g. For ordinary differential equations, Ward [18] used a sign condition for Duffing equations with no assumption on the growth of g, improving a result of Lazer [7]. Our main result improve a result of Haracek in [9], [10] when $n = 1$, and author's results in [11].

Several authors deal with the periodic–Dirichlet problem for this kind of non-linear dissipative hyperbolic equations or more generalized forms of non-linear dissipative hyperbolic equations. For example, Felmer and Manasevich [4], Haraux [6], Prodi [14], Biroli [2], Horácek [9], [10], Brezis and Nirenberg [3], Nkashama and Willem [13], and Rabinowitz [15], [16] discuss the existence of periodic–Dirichlet solutions for non-linear dissipative hyperbolic equations. Felmer and Manasevich, Haraux, Brezis and Nirenberg, and Nkashama and Willem allow g to grow at most linearly and Brezis and Nirenberg assume a Lendesman–Lazer type condition relating the forcing term to the non-linear term. Felmer and Manasevich, and Biroli assume a monotonicity condition on the non-linear term. Prodi imposes a Lipschitz condition on the non-linear forcing term. In Rabinowitz's work, he discusses equations having a non-linear term with parameter ϵ and his existence theorem quite depends on ϵ, for example, for sufficiently small ϵ. In our results, we have no need of such smallness conditions, and we allow g to grow superlinearly in u provided it satisfies a sign condition. The rate of growth allowed in g is arbitrary polynomial growth. We impose no monotonicity condition on g and we have no restriction on the forcing term h except that h is a measurable real valued Lebesgue square integrable function.

We derive an abstract realization of the linear dissipative hyperbolic differential operator and set up an associated abstract operator equation by using Fourier series. We represent the inverse of the linear hyperbolic operator by means of an integral operator with a kernel. We also note that the kernel of the linear operator of this equation is trivial ; i.e., the linear part is bijective. The compactness of this operator is treated and norm is also estimated.

Our proof is based on use of Fourier series and Leray–Schauder's continuation theorem. The use of Leray–Schauder's continuation theorem is based on the compactness of the inverse of the linear dissipative hyperbolic operator and bijectiveness of that linear dissipative hyperbolic operator. The main difficulty is to find an a'priori bound for all possible solutions of the associated non–linear equations. It is essential to our argument that $\beta \neq 0$.

2. Preliminary results

Now consider the equation

$$\beta u_t + u_{tt} - u_{xx} = h(t,x) \text{ where } \beta \neq 0,\ u = u(t,x). \tag{2.1}$$

If u and $h \in L^2(\Omega)$, we may write

$$u(t,x) = \sum_{(l,m)\in Z\times Z+} u_{lm} \exp(ilt)\sin(mx),$$

$$h(t,x) = \sum_{(l,m)\in Z\times Z+} h_{lm} \exp(ilt)\sin(mx)$$

with $\overline{u}_{lm} = u_{-lm}$ and $\overline{h}_{lm} = h_{-lm}$ since u and h are real.

The proof of the following lemma is clear and will be omitted.

LEMMA 2.1. *$u \in L^2(\Omega)$ is a weak solution to (2.1) if and only if*

$$u(t,x) = \sum_{(l,m)\in Z\times Z+} \left[\beta li + (m^2 - l^2)\right]^{-1} h_{lm} \exp(ilt)\sin(mx)$$

Let

$$\operatorname{Dom} L = \Big\{u \in L^2(\Omega) : \sum_{(l,m)\in Z\times Z+} \left[\beta^2 l^2 + (m^2 - l^2)^2\right] |u_{lm}|^2 < \infty\Big\}.$$

Define an operator $L : \operatorname{Dom} L \subseteq L^2(\Omega) \to L^2(\Omega)$ by

$$(Lu)(t,x) = \sum_{(l,m)\in Z\times Z+} \left[\beta li + (m^2 - l^2)\right] u_{lm} \exp(ilt)\sin(mx).$$

Then $\operatorname{Dom} L$ is dense in $L^2(\Omega)$, $\operatorname{Ker} L = \{0\}$, $\operatorname{Im} L = L^2(\Omega)$. Hence $L^{-1} : L^2(\Omega) \to \operatorname{Dom} L$ exists and

$$(L^{-1}h)(t,x) = \sum_{(l,m)\in Z\times Z+} \left[\beta li + (m^2 - l^2)\right]^{-1} h_{lm} \exp(ilt)\sin(mx).$$

Therefore, by lemma 2.1, if $h \in L^2(\Omega)$, then u is a weak solution of the periodic–Dirichlet problem on Ω for the equation

$$\beta u_t + u_{tt} - u_{xx} = h(t,x), \ \ \beta \neq 0 \text{ and } u = u(t,x),$$

if and only if $u \in \operatorname{Dom} L$, $Lu = h$, or $u = L^{-1}h$.

REMARK 2.1. *$L : \operatorname{Dom} L \subseteq L^2(\Omega) \to L^2(\Omega)$ is closed.*

LEMMA 2.2. *If $h \in L^2(\Omega)$ then there exists a constant $c > 0$ independent of h such that $\|L^{-1}h\|_\infty \leq c\|h\|_{L^2}$. The operator $L^{-1} : L^2(\Omega) \to C(\Omega)$ is compact.*

Proof. See [11], [12].

Combining the facts in [5], [8] and lemma 2.2, we have the following lemma.

LEMMA 2.3. *Dom*$L = L^{-1}(L^2(\Omega)) \subseteq W^{1.2}(\Omega) \cap C(\Omega)$ *and* $L^{-1}[W^{k.2}(\Omega)] \subseteq W^{k+1.2}(\Omega)$ *for* $k = 0, 1, 2, 3, \cdots$. *Moreover,* $\|L^{-1}h\|_{W^{1.2}} \leq C_1 \|h\|_{L^2}$ *where* $h \in L^2(\Omega)$ *and* C_1 (> 0) *is a constant independently of* h.

3. Main results

THEOREM. *Let* $h \in L^2(\Omega)$ *and suppose that*

(H_1) *is satisfied and*

(H_2) *for all* $(x, u) \in I \times R$, $ug(x, u) \geq -\psi(|u|)$ *where* $\psi : R \to R$ *is a function such that* $\limsup_{|u| \to \infty} \psi(|u|)/|u| = \alpha_0$.

Then the periodic–Dirichlet problem on Ω *for equation (1.1) has at least one weak solution.*

Proof. Now we see that L^{-1} ; $L^2(\Omega) \to C(\Omega)$ is a continuous and compact operator as a mapping into $C(\Omega)$. Define a substitution operator $N : C(\Omega) \to L^2(\Omega)$ by $Nu = -g(\cdot, u(\cdot, \cdot)) + h(\cdot, \cdot)$ for all $u \in C(\Omega)$. Then u is a weak solution of the periodic–Dirichlet problem for (1.1) if and only if $u \in$ Dom$L$ and satisfies

$$Lu = Nu, \text{ or equivalently} \tag{3.1}$$

$$u = L^{-1}Nu. \tag{3.2}$$

If $u \in C(\Omega)$ solves the operator equation (3.2), then $u \in C(\Omega)$ is a weak solution to the periodic–Dirichlet problem. Since L^{-1} is compact, and N is continuous and maps bounded sets into bounded sets, the composition $L^{-1}N : C(\Omega) \to C(\Omega)$ is continuous and compact. By using Leray–Schauder theory if all solutions u to the family of equations

$$u = \lambda L^{-1}Nu, \ 0 \leq \lambda \leq 1, \tag{3.3}$$

are bounded in $C(\Omega)$ independently of $\lambda \in [0, 1]$ then (3.1) has a solution. If (u, λ) solves (3.3), then (u, λ) solves

$$Lu = \lambda Nu \tag{3.4}$$

and u is a weak solution to the periodic–Dirichlet problem of the equation $\beta u_t + u_{tt} - u_{xx} + \lambda g(u) = \lambda h(t, x)$. Thus the proof will be completed if we show that the solutions to (3.4) for $0 \le \lambda \le 1$ are bounded in $C(\Omega)$ independently of $\lambda \in [0,1]$. Since, if $\lambda = 0$, we have only the trivial solution $u \equiv 0$, it suffices to show our assertion for $0 < \lambda \le 1$. To this end, let (u, λ) be any solution to (3.4) with $0 < \lambda \le 1$. By taking the inner product with u_t on the both sides of (3.4), we obtain

$$(Lu, u_t) + \lambda \iint_\Omega g(x, u(t,x)) u_t \, dt \, dx = \lambda \iint_\Omega h(t,x) u_t \, dt \, dx.$$

Since $Lu \in L^2(\Omega)$, there exists a sequence $\{y_n\} \subseteq C^\infty(\Omega)$ such that $y_n \to Lu$ in $L^2(\Omega)$ as $n \to +\infty$.

Let $u_n = L^{-1} y_n$. By lemma 2.3 and the Sobolev embedding theorems ; i.e., $W^{j+2,2}(\Omega) \hookrightarrow C^j(\Omega)$, $j = 0, 1, 2, \cdots$, $u_n \in C^\infty(\Omega)$ also. Since L^{-1} is continuous from $L^2(\Omega)$ into each of $W^{1.2}(\Omega)$ and $C(\Omega)$, we also have that $u_n \to L^{-1}(Lu) = u$ in each of these spaces as $n \to +\infty$. Thus $u_{nt} \to u_t$ in $L^2(\Omega)$. Now integration of these smooth functions, using the boundary conditions, shows that for each $n \in Z^+$, $(Lu_n, u_{nt}) = \beta \|u_{nt}\|^2_{L^2}$. Letting $n \to +\infty$ we obtain $(Lu, u_t) = \beta \|u_t\|^2_{L^2}$. Moreover, since for each $n \in Z^+$ the periodicity of $u_n(t,x)$ in t implies $(g(\cdot, u_n), u_{nt}) = 0$, we also obtain $(g(\cdot, u), u) = 0$. From this we can see that $Lu = \lambda N u$, $0 < \lambda \le 1$, implies

$$\beta \|u_t\|^2_{L^2} = \lambda (h, u_t)$$

and thus

$$\|u_t\|_{L^2} \le \frac{1}{|\beta|} \|h\|_{L^2}. \tag{3.5}$$

We next prove that $\|u\|_{L^2} \le M$ for some $M > 0$ independently of $\lambda \in]0,1]$.

Since $\limsup_{|u| \to \infty} \psi(|u|)/|u| = \alpha_0$, for $\alpha \ge 0$ with $\alpha > \alpha_0$, there exists $r_0 > 0$ such that $\psi(|u|)/|u| \le \alpha$ for all u with $|u| > r_0$. So $\psi(|u|) \le \alpha |u|$ for all u with $|u| > r_0$. Since g is Caratheodory function on $I \times R$, there exists a constant $M_{r_0} > 0$ such that $|g(x,u)| \le M_{r_0}$ for u with $|u| \le r_0$ for $x \in I$. Hence

$$\iint_{|u| \le r_0} u g(x,u) dt \, dx \le 2\pi^2 r_0 M_{r_0}$$

and thus

$$\iint_{|u|\le r_0} ug(x,u)dt\,dx \ge -2\pi^2 r_0 M_{r_0}.$$

Hence

$$\iint_{\Omega} ug(x,u)dt\,dx = \iint_{|u|>r_0} ug(x,u)dt\,dx + \iint_{|u|\le r_0} ug(x,u)dt\,dx$$
$$\ge -2^{\frac{1}{2}}\alpha\pi\|u\|_{L^2} - 2\pi^2 r_0 M_{r_0}.$$

By taking the inner product with u on each side of (3.4), and an argument similar to that used to establish (3.5) shows

$$(Lu,u) = \|u_x\|_{L^2}^2 - \|u_t\|_{L^2}^2.$$

Thus,

$$\|u_x\|_{L^2}^2 - \|u_t\|_{L^2}^2 + \lambda\iint_{\Omega} ug(x,u)dt\,dx = \lambda\iint_{\Omega} h(t,x)u\,dt\,dx.$$

Hence, for $0<\lambda\le 1$ we have, by (3.5)

$$\|u_x\|_{L^2}^2 = \|u_t\|_{L^2}^2 - \lambda\iint_{\Omega} ug(u)dt\,dx + \lambda\iint_{\Omega} h(t,x)u\,dt\,dx$$
$$\le \left[2^{\frac{1}{2}}\alpha\pi + \|h\|_{L^2}\right]\|u\|_{L^2} + \frac{1}{|\beta|^2}\|h\|_{L^2}^2 + 2\pi^2 r_0 M_{r_0}.$$

But since $\|v\|_{L^2}\le C_2\|v_x\|_{L^2}$ for all $v\in \mathrm{Dom}L$,

$$\|u\|_{L^2}^2 \le C_2^2\left[(2^{\frac{1}{2}}\alpha\pi + \|h\|_{L^2})\|u\|_{L^2} + \frac{1}{|\beta|^2}\|h\|_{L^2}^2 + 2\pi^2 r_0 M_{r_0}\right].$$

Therefore, there exists a constant M_1 independently of $\lambda\in\,]0,1]$ such that $\|u\|_{L^2}\le M_1$.

Again, let (u,λ) be any solution for (3.4). By taking the inner product with u on the both sides of (3.4), we have

$$\|u_x\|_{L^2}^2 - \|u_t\|_{L^2}^2 + \iint_{\Omega} ug(x,u)dt\,dx \le \|h\|_{L^2}\|u\|_{L^2}$$

again since

$$\iint_\Omega ug(x,u)dt\,dx \geq -\left[2^{\frac{1}{2}}\alpha\pi M_1 + 2\pi^2 r_0 M_{r_0}\right],$$

$$\begin{aligned}\|u_x\|_{L^2}^2 &\leq \|u_t\|_{L^2}^2 + \|h\|_{L^2}\|u\|_{L^2} + 2^{\frac{1}{2}}\alpha\pi M_1 + 2\pi^2 r_0 M_{r_0} \\ &\leq \frac{1}{|\beta|^2}\|h\|_{L^2}^2 + \left[2^{\frac{1}{2}}\alpha\pi + \|h\|_{L^2}\right]M_1 + 2\pi^2 r_0 M_{r_0}.\end{aligned}$$

Thus, $\|u_x\|_{L^2} \leq M_2$ for some $M_2 > 0$ independently of $\lambda \in]0,1]$. Therefore, we have $u \in W^{1.2}(\Omega)$ and $\|u\|_{W^{1.2}} \leq L_1$ where L_1 is a constant that may depend on $\beta, h, M_1, M_2, \alpha, r_0, M_{r_0}$ but is independent of $\lambda \in]0,1]$. Since $L^2(\Omega) \subseteq L^q(\Omega)$ where $1 \leq q \leq 2$ and since $W^{1.2}(\Omega)$ is embedded in $L^q(\Omega)$ where $2 \leq q < \infty$ (see, e.g., [1], [17]), $\|u\|_{L^q} \leq L_2(q)$ where L_2 may depend on L_1 and $q \geq 1$ but is independent of $\lambda \in]0,1]$.

Next, we will estimate the L^2–bound for $g(\cdot,u)$.

For $|u| \leq d_0 + 1,\ x \in I$, we have

$$|g(x,u)| \leq \sup_{\substack{x\in I \\ |u|\leq d_0+1}} |g(x,u)| \leq M_3$$

since g is a Caratheodory function,and for $|u| \geq d_0 + 1,\ x \in I$, we have

$$|g(x,u)| = 1/|u|\ |ug(x,u)| \leq 1/d_0 + 1\big(a(x)|u|^{p+1} + b(x)|u|\big).$$

Therefore, we have

$$\begin{aligned}|g(x,u)| &\leq \sup_{\substack{x\in I \\ |u|\leq d_0+1}} |g(x,u) + 1/|u|\ |ug(x,u)| \\ &\leq 1/d_0 + 1(a(x)|u|^{p+1} + b(x)|u|) + M_3.\end{aligned}$$

Hence

$$\begin{aligned}\|g(\cdot,u)\|_{L^2}^2 \leq M_4\|u\|_{L^{2p+2}}^{2p+2} + M_5\|u\|_{L^{p+2}}^{p+2} + M_6\|u\|_{L^{p+1}}^{p+1} \\ + M_7\|u\|_{L^2}^2 + M_8\|u\|_{L^i} + M_9\end{aligned}$$

for some appropriate constants $M_4, \cdots, M_9$. Since $\|u\|_{L^q} \leq L_2(q)$ for $q \geq 1$, $\|g(\cdot, u)\|_{L^2} \leq L_0$ for some L_0 where L_0 may depends only on $M_4, \cdots, M_9$ and $p > 0$. So if (u, λ) is any solution to (3.3), then using Lemma 2.3.,

$$\begin{aligned} \|u\|_\infty = \lambda \|L^{-1} Nu\|_\infty &\leq C(\|g(\cdot, u)\|_{L^2} + \|h\|_{L^2}) \\ &\leq C(L_0 + \|h\|_{L^2}) \end{aligned}$$

and this completes our proof.

Now, we define $g(x, \pm\infty)$ by $g(x, +\infty) = \liminf_{u \to \infty} g(x, u)$ and $g(x, -\infty) = \limsup_{u \to \infty} g(x, u)$, where $g(x, +\infty)$, $g(x, -\infty)$ are in $L^\infty(I)$ and hold uniformly in the following sense ;

For any $\epsilon > 0$, and $s \in L^\infty(I)$ with $g(x, +\infty) > s(x)$ for all $x \in I$, there is $r_0 > 0$ such that for all $x \in I$, and $u \geq r_0$, $g(x, u) + \epsilon > s(x)$. And if $g(x, -\infty) < s(x)$, then $g(x, -u) - \epsilon < s(x)$ for all $x \in I$, and $u \geq r_0$.

Corollary. *Let $h \in L^2(\Omega)$ and suppose that*

(H_1) *is satisfied and*

(H_3) $g(x, -\infty) < g(x, +\infty)$ *and* $g(x, +\infty) - g(x, -\infty)$ *has a positive infimum on I. Then the periodic–Dirichlet problem on Ω for the equation (1.1) has at least one weak solution.*

Proof. By the definition of $g(x, \pm\infty)$ and (H_3), there exist $r_0 > 0$ and $s(x)$ in $L^\infty(I)$ such that $g(x, -u) \leq s(x) \leq g(x, +u)$ for all $|u| \geq r_0$ and all $x \in I$. If we define $\tilde{h}$ by $h(t, x) - s(x)$ and $\tilde{g}$ by $g(x, u) - s(x)$, then $\tilde{g}$ still satisfies the growth condition and $\tilde{g}(x, -u) \leq 0 \leq \tilde{g}(x, u)$ for all $|u| \geq r_0$ and all $x \in I$. Thus $u\tilde{g}(x, u) \geq 0$ for all $|u| \geq r_0$ and for all $x \in I$ and this proves our corollary.

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Dong–A University
Pusan 604–714, Korea