Comm. Korean Math. Soc. 4(1989), No. 1, pp. 33~46

ASYMPTOTIC DECAYS OF PAIR CORRELATION FUNCTIONS FOR SELF-AVOIDING RANDOM SURFACES *

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1. Introduction

In this paper, we study the truncated pair correlation functions for a class of self-avoiding random cylinder surfaces in d-dimensional lattice space Z^d. There has been considerable interest in mathematical theories of random surfaces [1,3,4,6]. Such theories appear to play an important role in quantum field theory, statistical physics and percolation theory [4]. Recently Abraham, Chayes and Chayes [1] have studied the truncated pair functions for the solid-on-solid surfaces. Our main purpose is to extend their results to a more wider class of surfaces.

The correlation functions of lattice gauge theories, three-dimensional spin systems and models of crystalline interfaces have natural expression as weighted sums over surfaces. However, such expressions are difficult to analyze due to both the combinatoric problems introduced by large number of surfaces, and the intractability of explicit forms for the associated weights (See [3] and the references therein). It is therefore of interest to study models of correlation functions which are defined as sums over restricted classes of surfaces with relatively simple weights.

We analyze the behaviour of correlation functions of the form

$$Q_C(\beta) = \sum_{S \in C} e^{-\beta|S|} \tag{1.1}$$

where β is a positive parameter (the inverse temperature). C denotes some prescribed class of surfaces on the lattice Z^d, and $|S|$ is the area

Received September 21, 1988.
*Work supported in part by the Korean Science Foundation and the Basic Science Research Program, Korean Ministry of Education, 1988.

(i.e., the number of plaquettes (unit squares)) of the surfaces $S \in C$. Our principal results concern the truncated pair function (i.e. glue-ball propagators), $Q_{C_L}(\beta) \equiv Q_L(\beta)$. Here, $C = C_L$ denotes some set of tubular surfaces which have as their boundary the edges of the two plaguettes (elementary unit squares) separated by a distance L lattice units along x_1-axis. These surfaces defined only on $\frac{1}{2} \leq x_1 \leq L+\frac{1}{2}$ so we called it *cylinder model.* The quantity $Q_L(\beta)$ serves as an approximation to the low temperature expansion of the truncated pair correlation in a three-dimensional ferromagnet. Such random surface approximations are quite accurate in the low temperature regime [5].

Analogues of $Q_L(\beta)$ may be defined for classes of surface other than C_L. For a subclass S_L of C_L, solid-on-solid model have been studied in detail by Abraham, Chayes and Chayes [1]. Indeed, in this paper, we extend the restricted class of surface of [1]. By improving the methods used in [1], we will get the results similar to those in [1].

In Section 2, we show that for sufficiently large β

$$Q_L(\beta) \sim \frac{1}{L^{\frac{(d-1)}{2}}} e^{-M(\beta)L} \quad \text{as } L \to \infty \tag{1.2}$$

for the constrained surfaces where $M(\beta)$ is the glue-ball mass or the inverse correlation length. Two ingredients are necessary for the proof of (1.2). First we show that $M(\beta)$ exists and is strictly positive above some melting point β_c. We then develop a random surface Ornstein-Zernike equation which enables us to establish the power law correlations indicated in (1.2). In Section 3, we extend the result to the unconstrained surfaces. Our main results are stated in Theorm 2.4 and Theorem 3.5 respectively.

2. Asymptotic decay of cylinder pair correlation for constrained surfaces

In this section we study a truncated pair correlation of the form (1.1) for the subclasses of surfaces of C_L and C'_L defined below. The principal result (Theorem 2.4) of this section is a proof of Ornstein-Zernike scaling (Eq.(1.2)) of the pair function for constrained surfaces.

Our method relies on the approach initiated in [1] which shows that the original ideas of Ornstein-Zernike [8] may be implemented whenever one can define a directed correlation function with a strictly large decay rate than that of the two point function (see [1] for the details). For clarity of exposition, throughout the analysis we restrict our attention to the three– dimensional cubic lattice Z^3. The surfaces in C_L and C'_L will be constructed from plaquettes (unit squares: 2–cells) on the dual lattice $\{x + (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) : x \in Z^3\}$ of Z^3. We denote by p_0 the edges of the plaquette centered at $(\frac{1}{2}, 0, 0)$ and by p_L the translation of p_0 through L units in the x–direction. The plane $x = k$ will be denoted by P_k.

DEFINITION. *C'_L is the set of all connected, self–avoiding surfaces S with boundary $\partial S = p_0 \cup p_L$, satisfying the condition that the intersection of S with each of the plane P_k, $1 \leq k \leq L$ is a set of disjoint closed curves $\gamma_k \equiv S \cap P_k$ and $\gamma_k \equiv \emptyset$ for $k < 1$ or $k > L$.*

Thus any surface in C'_L can have overhangs and handles. For convenience, we will study the subclass $C_L \subset C'_L$ of surfaces for which the sets of closed curves, γ_1 and γ_L, are required to be the only one elementary square. In other words, the surfaces in C_L begin and end on open boxes of four plaquettes surrounding the points (1,0,0) and $(L, 0, 0)$, respectively. Such boxes will be called *elementary chimneys*. The restriction of C_L causes no loss of generality, since the pair correlations $Q_L = \sum_{S \in C_L} e^{-\beta |S|}$ and $Q'_L = \sum_{S \in C'_L} e^{-\beta |S|}$ are related by

$$Q_{L+2} = e^{-8\beta} Q'_L \tag{2.1}$$

In this section, we will be concerned with surfaces in $c_L \subset C_L$ which satisfy the additional restriction that among each of the set of rings γ_k, there is a ring γ_k^0 which surrounds the origin of P_k. For obvious reason, this subclass will be called the set of *constrained surfaces*. The associated constrained pair correlation will be denoted by q_L.

We now establish some elementary results on the asymptotic behaviour of $Q_L(\beta)$ for large L.

PROPOSITION 2.1. *For all β*

$$\lim_{L \to \infty} \frac{log Q_L(\beta)}{L} \equiv -M(\beta) \tag{2.2}$$

exists (in the extended real line).

Proof. This follows from subadditivity [5]. Indeed, those surfaces composed of a tube in C_{L_1}, joined to a tube in C_{L_2} (which has been translated L_1 units in the x-direction) form a subset of $C_{L_1+L_2}$. Thus

$$Q_{L_1+L_2} \geq Q_{L_1} Q_{L_2} \tag{2.3}$$

and so $-\log Q_L(\beta)/L$ satisfies the subadditivity.

COROLLARY. *$-M(\beta)$ provides a uniform upper bound on $\log Q_L/L$, i.e.*

$$Q_L(\beta) \leq e^{-M(\beta)L} \quad \forall L. \tag{2.4}$$

In particular, since $Q_L(\beta) \geq e^{-4\beta L}$,

$$M(\beta) \leq 4\beta. \tag{2.5}$$

PROPOSITION 2.2. *The mass $M(\beta)$ is a concave and nondecreasing function of β. Furthermore, $\exists\ 0 < \beta_C < \infty$ such that $M(\beta) > 0$, for all $\beta > \beta_C$ and $M(\beta) < 0, \forall \beta < \beta_C$.*

Proof. The result follows from Proposition 2.1 and the above corollary. For the details, see [4].

REMARK. *The concavity of $M(\beta)$ implies that it can have at most a single jump discontinuity. Should this occurs at some $\overline{\beta}$, then $M(\beta) = -\infty$ for all $\beta < \overline{\beta}$.*

We note that Propositions 2.1 and 2.2 are basic consequences of the form of the pair correlation (1.1), and are not sensitive to the class of surfaces under consideration, provided they are subadditive. In particular, the results obviously hold for constrained cylinder surfaces.

We will now establish that the pair correlation $q_L(\beta)$ for constrained surfaces decays via a pure exponential. In order to prove the result, we introduce the direct correlation function. Recall that the surfaces in c_L must begin (and end) on an elementary chimney. We may classify the surfaces in c_L according to the location of their next elementary chimney. Consider, then, the correlation function $d_L(\beta)$ obtained by summing over surfaces $t_L \subset C_L$ which contains no elementary chimneys other than the first and last;

$$d_L(\beta) = \sum_{S \in t_L} e^{-\beta |S|} \tag{2.6}$$

$$t_L = \{S \in C_L : |\gamma_k| > 4 \text{ for all } k \text{ except } k = 1, L\} \tag{2.7}$$

The following proposition shows that d_L should have a shorter range (i.e. a larger mass) than q_L.

PROPOSITION 2.3. *For all β,*

$$\lim_{L \to \infty} \frac{\log d_L(\beta)}{L} \equiv -m_d(\beta) \tag{2.8}$$

exists. Moreover, for sufficiently large β, $m_d(\beta) > m(\beta)$, where $m(\beta)$ is the mass of constrained surfaces.

Proof. The limit exists from another subadditivity estimate of the form

$$d_{L_1+L_2}(\beta) \geq (const) d_{L_1}(\beta) d_{L_2}(\beta), \tag{2.9}$$

where the constant is independent of the lengths L_1 and L_2.

We may bound $m_d(\beta)$ below by, say, a Peierl's argument: For a given plaquette there are 11 ways to attach another plaquette and so

$$|\{S \in t_L : |S| = n\}| \leq 11^n = e^{n \log 11}$$

which implies

$$d_L \leq \sum_{n=6L}^{\infty} e^{-n(\beta - \log 11)}$$

and so

$$m_d(\beta) \geq 6(\beta - \log 11) \tag{2.10}$$

for $\beta > \log 11$. Comparing this with the a priori upper bound on $m(\beta)$ given by (2.5), the result is seen to hold for β large enough.

THEOREM 2.4. *Whenever $m_d(\beta) > m(\beta)$, $\exists\ K_1(\beta), K_2(\beta) > 0$ such that*

$$|q_L(\beta)e^{m(\beta)L} - K_1(\beta)| \leq e^{-K_2(\beta)L} \tag{2.11}$$

uniformly in L.

Proof. Partitioning the set c_L according to the scheme outlined above, it is seen that the contribution from all surfaces which do not have an elementary chimney until the N^{th} step $(2 \leq N \leq L-1)$ is given by

$$\frac{1}{g^4} d_N q_{L+1-N}, \qquad g \equiv e^{-\beta}. \tag{2.12}$$

We may therefore write

$$q_L = d_L + \frac{1}{g^4} \sum_{N=2}^{L-1} d_N q_{L+1-N}. \tag{2.13}$$

To complete the argument, we will exploit the fact that (2.18) is of the form of a convolution. Consider the discrete Laplace transform

$$\hat{q}(z) = \sum_L q_L z^L \tag{2.14}$$

$$\hat{d}(z) = \sum_L d_L z^L \tag{2.15}$$

Here we define $q_1 = d_1 = g^4$. Evidently $\hat{q}(z)$ is analytic in the region $|z| < e^m$, while $\hat{d}(z)$ admits the larger region of analyticity $|z| < e^{m_d}$.

Taking the transform of Eq. (2.16), a little algebra yields

$$\hat{q}(z) = \frac{zg^4}{2 - \frac{1}{zg^4}\hat{d}(z)} \tag{2.16}$$

for $|z| < e^m$. Note however that the right-hand side of (2.16) makes sense in the larger region $|z| < e^{m_d}$. Indeed using the bound (2.4), and nonnegativity of d_L it is easy to show that the function $K(z) = 2 - \frac{1}{zg^4}\hat{d}(z)$ has a simple zero at $z = e^m$ and no other zeroes within some larger disk $|z| < e^m/\lambda$, $\lambda < 1$. Thus $zg^4/K(z)$ defines a meromorphic extension for $\hat{q}(z)$ in the region $|z| < e^m/\lambda$ with a simple pole at $z = e^m$. We may therefore write

$$\hat{q}(z) = \frac{F(z)}{(1 - ze^{-m})} \tag{2.17}$$

with $F(z)$ analytic for $|z| < e^m/\lambda$. Noting that q_L is simply the coefficient of z^L in the expansion of the above equation, we obtain

$$q_L e^{mL} = F_0 + F_1 e^m + \cdots + F_L e^{mL} \tag{2.18}$$

Finally, recalling the Cauchy bound

$$|F_n| \leq (const)\lambda^n e^{-mn} \tag{2.19}$$

the desired result follows easily.

3. Ornstein–Zernike decay for unconstrained surfaces

We now treat the case where the surface is permitted to wander from the x–axis. The main idea we will use is essentially same as that used in the proof of Theorem 2.7 of [1]. To make the paper self–contained, as much as possible, we will produce the proofs in details. In order to facilitate our analysis, we introduce a generalization of the correlation $Q_L(\beta)$, which we denote by $Q_{L,(a,b)}(\beta)$. The latter function is defined by summing over all cylinder tubes which begin and end in elementary chimneys, and have as their boundary $p_0 \cup p_{L,(a,b)}$. Here $p_{L,(a,b)}$ denote the translate of p_L by a unit in the y–and b units in the z–direction. Thus $Q_L = Q_{L,(0,0)}$. Finally, we also define the *master function*, $\mathbf{Q}$, to be the sum over all cylinder tubes which begin at p_0 and end somewhere in the plane $x = L + \frac{1}{2}$, i.e.

$$\mathbf{Q}_L(\beta) = \sum_{(a,b)} Q_{L,(a,b)}(\beta). \tag{3.1}$$

All of the above have direct correlation counterparts ; $D_{L,(a,b)}$, $D_L = D_{L,(0,0)}$ and $\mathbf{D}_L = \sum_{(a,b)} D_{L,(a,b)}$.

As will become apparent, the master functions $\mathbf{Q}_L$ and $\mathbf{D}_L$ behave similarly to the constrained correlations q_L and d_L discussed in subsection (i). We shall exploit this analogy to prove the desired scaling for $Q_L(\beta)$. First we note that, by standard subadditivity arguments, the master function has well–defined masses:

PROPOSITION 3.1. *The limits*

$$M(\beta) \equiv \lim_{L\to\infty} [-log\, \mathbf{Q}_L(\beta)/L] \tag{3.2}$$

and

$$M_d(\beta) \equiv \lim_{L\to\infty} [-log\, \mathbf{D}_L(\beta)/L] \tag{3.3}$$

exist (in the extended real line).

REMARK. *Our use of $M(\beta)$ to denote the limit in (2.2) and that in (3.2) will be justified by the latter results (Theorem 3.5).*

Next, following the argument of Eqs. (2.12)–(2.13), we observe that the correlations are related by an Ornstein–Zernike equation:

$$Q_{L,(a,b)} = D_{L,(a,b)} + \frac{1}{g^4}\sum_{N=2}^{L-1}\sum_{a',b'} D_{N,(a',b')}Q_{L+1-N,(a-a',b-b')} \tag{3.4A}$$

$$\mathbf{Q}_L = \mathbf{D}_L + \frac{1}{g^4}\sum_{N=2}^{L-1} \mathbf{D}_N \mathbf{Q}_{L+1-N}. \tag{3.4B}$$

Equation (3.4B) may be obtained simply by summing (3.4A) over a and b.

The relevant transforms are given by

$$\hat{Q}_\beta(z,(\omega_1,\omega_2)) = \sum_{L,(a,b)} Q_{L,(a,b)}(\beta) z^L e^{i\omega_1 a} e^{i\omega_2 b} \tag{3.5A}$$

$$\hat{\mathbf{Q}}_\beta(z) = \sum_L \mathbf{Q}_L(\beta) z^L = \hat{Q}_\beta(z,(0,0)) \tag{3.5B}$$

and similarly for D. In the above, z is a complex number (presumably of modulus smaller than $e^{M(\beta)}$), and $-\pi < \omega_1, \omega_2 \leq \pi$. As before, we use the convolution form of (3.4A) to write

$$\hat{Q}(z,(\omega_1,\omega_2)) = \frac{zg^4}{(2 - \hat{D}(z,(\omega_1,\omega_2))/zg^4)}. \tag{3.6}$$

The corresponding equation for the master function is obtained by setting $\omega_1 = \omega_2 = 0$.

We regain the quantities Q_L by means of the inversion formula

$$Q_L = \frac{1}{2\pi_i} \oint \frac{dz}{z^{L+1}} \int_{-\pi}^{\pi} d\omega_1 d\omega_2 \hat{Q}(z,(\omega_1,\omega_2)). \tag{3.7}$$

However, in order to analyze the above integral, we must establish that $\hat{Q}$ and $\hat{D}$ have certain continuity properties. In this paper, it is convinient to regard $\xi_1 = e^{i\omega_1}$ and $\xi_2 = e^{i\omega_2}$ as complex variables restrict to the unit circle. Although a weaker result would suffice for the purpose of this section, we show below that those values of z at which the transformed correlation functions are (separately) regular in ξ_1 and ξ_2 in a neighborhood of $|\xi_1| = 1$ and $|\xi_2| = 1$. We state the lemma for direct functions, the analogous result holds for the $Q's$.

LEMMA 3.2. *Take $|z| < e^{M_d(\beta)}$, $\beta > \overline{\beta}$. Then, provided that $|\xi_1|$ and $|\xi_2|$ are sufficiently close to one, the function*

$$\hat{D}_\beta(z,\xi_1,\xi_2) = \sum_{L,a,b} D_{L,(a,b)} z^L \xi_1^a \xi_2^b \tag{3.8}$$

is regular in $\xi_1(\xi_2)$ *for fixed* $\xi_2(\xi_1)$.

Proof. See [1].

For latter use, we analyze the integral

$$Q_{L,(a,b)} = \frac{1}{2\pi_i} \oint \frac{dz}{z^{L+1}} \int_{-\pi}^{\pi} d\omega_1 d\omega_2 e^{-ia\omega_1} e^{-ib\omega_2} \frac{zg^4}{2 - \frac{1}{zg^4} \hat{D}(z, (\omega_1, \omega_2))}. \tag{3.9}$$

We first demonstrate that as $L \to \infty$, the only significant contribution to the integral is from an infinitesimal neighborhood of $\omega_1 = \omega_2 = 0$.

LEMMA 3.3. $\forall \delta > 0$, $\exists \epsilon, \nu > 0$ *such that unless* $|\omega_1| = |\omega_2| < \delta$

$$|2 - \frac{1}{zg^4} \hat{D}(z, (\omega_1, \omega_2))| > \frac{\nu |z|^2}{g^4} \tag{3.10}$$

$\forall z$ *with* $|z| \leq e^{M+\epsilon}$.

Proof. It is convenient to express $\hat{D}(z, (\omega_1, \omega_2))$ as a power series in z with coefficients $D_L(\omega_1, \omega_2)$

$$\begin{aligned} \hat{D}(z, (\omega_1, \omega_2)) &= \sum_{L,(a,b)} D_{L,(a,b)} e^{ia\omega_1} e^{ib\omega_2} z^L \\ &= \sum_L D_L(\omega_1, \omega_2) z^L \end{aligned} \tag{3.11}$$

$$(\text{i.e. } D_L(\omega_1, \omega_2) = \sum_{a,b} D_{L,(a,b)} e^{ia\omega_1} e^{ib\omega_2})$$

By the $a \leftrightarrow -a$, $a \leftrightarrow -b$ symmetry, each $D_L(\omega_1, \omega_2)$ is of the form of cosine series with positive coefficients, and hence is maximized by $\omega_1 = \omega_2 = 0$. Thus, given $\delta > 0$, there exists $\nu > 0$ such that

$$\sup_{\substack{|\omega_1| > \delta \\ |\omega_2| > \delta}} |D_3(\omega_1, \omega_2)| < D_3(0,0) - 2\nu\ . \tag{3.12}$$

If follows that, for $|\omega_1|$, $|\omega_2| > \delta$

(3.13)
$$\begin{aligned}|\hat{D}(z,(\omega_1,\omega_2))| &\leq \mathbf{D}_1|z| + \mathbf{D}_2|z|^2 + |D_3(\omega_1,\omega_2)|\,|z|^3 + \mathbf{D}_4|z|^4 + \cdots \\ &= \hat{\mathbf{D}}(|z|) + |z|^3[D_3(\omega_1,\omega_2) - D_3(0,0)] \\ &\leq \hat{\mathbf{D}}(|z|) - 2\nu|z|^3.\end{aligned}$$

Next, we find an $\epsilon > 0$ small enough so that

$$\left.\frac{\hat{\mathbf{D}}(x)}{xg^4}\right|_{x=e^{M+\epsilon}} < 2 + \frac{\nu e^{2M}}{g^4} \tag{3.14}$$

[Recall that $\hat{\mathbf{D}}(e^M)/e^M g^4 = 2$ and that $[\hat{\mathbf{D}}(x)/x]' > 0$ for x real and positive.] Since the coefficients, $\mathbf{D}_L$, are non–negative

$$|\frac{\hat{\mathbf{D}}(z)}{zg^4}| \leq \frac{\hat{\mathbf{D}}(|z|)}{|z|g^4} < 2 + \frac{\nu e^{2M}}{g^4} \tag{3.15}$$

whenever $e^M \leq |z| \leq e^{M+\epsilon}$. When $|z| < e^M$, we have the (stronger) bound.

$$|\frac{\hat{\mathbf{D}}(z)}{zg^4}| \leq \frac{\hat{\mathbf{D}}(|z|)}{|z|g^4} < 2 \tag{3.16}$$

(3.17)
$$\begin{aligned}|2 - \frac{1}{zg^4}\hat{D}(z,(\omega_1,\omega_2))| &\geq 2 - \frac{\hat{\mathbf{D}}(|z|)}{|z|g^4} + \frac{2\nu|z|^2}{g^4} \\ &\geq \frac{2\nu|z|^2}{g^4} - \chi(|z| \geq e^M)\nu e^{2M}\frac{\nu e^{2M}}{g^4} \\ &\geq \frac{\nu|z|^2}{g^4}\end{aligned}$$

In the above $\chi(|z| \geq e^M) = 1$ if $|z| \geq e^M$, and zero otherwise.

COROLLARY.

$$(3.18)\quad \left|\frac{1}{2\pi_i}\int_C \frac{1}{z^{L+1}}dz \int_{|\omega_1|>\delta} d\omega_1 \int_{|\omega_2|>\delta} d\omega_2 \frac{zg^4}{2-\frac{1}{zg^4}\hat{D}(z,(\omega_1,\omega_2))}\right| \leq \frac{(2\pi)^2}{\nu} g^8 e^{-(M+\epsilon)(L+1)}$$

where $C : |z| = e^{M+\epsilon}$.

LEMMA 3.4. *For any $A > 0$ satisfying $e^M + A < e^M/\lambda$, $\exists\,\delta > 0$ such that for all $|\omega_1|$, $|\omega_2| < \delta$, $[2 - \hat{D}(z,(\omega_1,\omega_2))/zg^4]$ and $[2 - \hat{\mathbf{D}}(z)/zg^4]$ have the same number of zeroes inside the disk $|z| < e^M + A$.*

Proof. Define $k = \min_{|z|=e^M+A} |2 - \hat{\mathbf{D}}(z)/zg^4| > 0$. By choosing L_0 and K_0 sufficiently large, we may ensure that

$$(3.19)\quad \sum_{L>L_0} \mathbf{D}_L(e^M+A)^L + \sum_{\substack{L>L_0\\ |a|,|b|>K_0}} D_{L,(a,b)}(e^M+A)^L < \frac{k}{3}(e^M+A)g^4$$

An analogous estimate holds for nonzero ω_1 and ω_2. Thus, on the circle $|z| = e^M + A$, we have

$$(3.20)\quad \left|\frac{1}{zg^4}\hat{\mathbf{D}}(z) - \frac{1}{zg^4}\hat{D}(z,(\omega_1,\omega_2))\right| < \frac{2}{3}k + \frac{1}{zg^4}\sum_{\substack{L<L_0\\ |a|,|b|<K_0}} D_L,(a,b)z^L(1-e^{ia\omega_1}e^{ib\omega_2})$$

Since the second term on the right–hand side of the above is a finite sum, we may find a $\delta > 0$ so that it is bounded above by $\frac{k}{3}$ whenever $|\omega_1|$, $|\omega_2| < \delta$. Applying the theorem of Rouché, the desired result follows.

We can now establish the Ornstein–Zernike scaling.

THEOREM 3.5. *Whenever $M(\beta) < M_d(\beta)$, Q_L has the asymptotic form*

$$(3.21)\quad Q_L(\beta) = \frac{(const)}{L} e^{-M(\beta)L}[1 + O(\frac{1}{\sqrt{L}})]$$

Proof. $\hat{\mathbf{Q}}(z)$ has a simple pole at $z = e^M$ and no other poles in the larger disk $|z| < e^M/\lambda$. This is done by following the steps for the proof of Theorem 2.4.

Next, a Rouché argument (lemma 2.8) established that when ω_1 and ω_2 are sufficiently small, $\hat{Q}(z,(\omega_1,\omega_2))$ has a simple pole and no other poles in the larger disk $|z| < e^M + A < e^M/\lambda$. Since $\hat{D}(z,(\omega_1,\omega_2))$ is smooth (indeed, analytic) in (ω_1,ω_2), there is a function $e^{M(\omega_1,\omega_2)}$ which describes the motion of simple pole for small (ω_1,ω_2), by the implicit function theorem. This pole is, of course, the principle contribution to the integral (3.7) for (ω_1,ω_2) sufficiently small. Therefore we have

$$(3.22)\quad Q_L(\beta) = \int_{|\omega_1|,|\omega_2|<\delta} d\omega_1 d\omega_2 e^{-LM(\omega_1,\omega_2)} F(\omega_1,\omega_2)\left[1 + O(e^{-\epsilon L})\right].$$

In the above, δ is some suitably chosen constant which ensures that both the Rouché argument in lemma 3.4 and the use of the implicit function theorem are legitimate. The function $F(\omega_1,\omega_2)$ is continuous in (ω_1,ω_2) and independent of L. Because all the coefficients $D_{L,(a,b)}$ are non–negative and symmetric in $a \leftrightarrow -a$, $a \leftrightarrow -b$, it follows that

$$(3.23)\qquad e^{M(\omega_1,\omega_2)} = e^{M+\gamma(\omega_1^2+\omega_2^2)+\cdots}$$

(That the coefficients of ω_1^2 and ω_2^2 are identical follows from the $a \leftrightarrow b$ symmetry.) The standard asymptotic analysis may now be applied to the ω integrations from which the desired result follows easily.

REMARK. *The analysis above is easily extended to cylinder tubes in higher dimensions. It is easy to see that each extra dimension provides an additional transverse degree of freedom, and thus another factor of* $\frac{1}{\sqrt{L}}$ *in the generalization of theorem 3.5. One must, of course, modify the constants in the bounds which guarantee a region where* $M(\beta) < M_d(\beta)$.

Acknowledgements. We are very grateful to Korean Science Foundation and Korean Ministry of Education for financial supports.

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