Comm. Korean Math. Soc. 4(1989), No. 1, pp. 17~22

ON QUOTIENT SEMIRING AND EXTENSION OF QUOTIENT HALFRING *

HEE SIK KIM

1. Introduction

Semiring was first introduced by H.S.Vandiver in 1934. Allen [1] introduced the notion of Q–ideal and constructed the quotient structure of a semiring modulo a Q–ideal. L.Dale [2] studied some relations between ideals in the halfring and ideals in the corresponding ring. With this concept we study some properties of quotient semiring and extension of quotient halfring. Definitions and theorems in [1], [4] and [7] are used in this paper.

DEFINITION 1.1 [1]. *A set R together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$, respectively) will be called a semiring provided :*

(1) *addition is a commutative operation,*
(2) *there exists $0 \in R$ such that $x + 0 = x$ and $x0 = 0x = 0$ for each $x \in R$, and*
(3) *multiplication distributes over addition both from the left and from the right.*

DEFINITION 1.2 [1]. *A subset I of a semiring R will be called an **ideal** if $a, b \in I$ and $r \in R$ implies $a + b \in I, ra \in I$ and $ar \in I$.*

DEFINITION 1.3 [1]. *An ideal I in the semiring R will be called a Q–ideal if there exists a subset Q of R satisfying the following conditions:*

(1) *$\{q + I\}_{q \in Q}$ is a partition of R ; and*
(2) *if $q_1, q_2 \in Q$ such that $q_1 \neq q_2$, then $(q_1 + I) \cap (q_2 + I) = \emptyset$.*

Received September 6, 1988 and in revised form February 24, 1989.
*This research was supported by MOE, 1987.

LEMMA 1.4 [1]. *Let I be a Q–ideal in the semiring R. If $x \in R$, then there exists a unique $q \in Q$ such that $x + I \subset q + I$.*

Let I be a Q–ideal in the semiring R. In view of the above results, one can define the binary operations $\oplus_Q$ and $\odot_q$ on $\{q + I\}_{q \in Q}$ as follows :

(1) $(q_1 + I) \oplus_Q (q_2 + I) = q_3 + I$ where q_3 is the unique element in Q such that $q_1 + q_2 + I \subset q_3 + I$; and
(2) $(q_1 + I) \odot_Q (q_2 + I) = q_3 + I$ where q_3 is the unique element in Q such that $q_1 q_2 + I \subset q_3 + I$.

THEOREM 1.5 [1]. *If I is a Q–ideal in the semiring R, then*

$$(\{q + I\}_{q \in Q},\ \oplus_Q,\ \odot_Q)$$

is a semiring and denoted by $\frac{R}{I}$.

THEOREM 1.6 [7]. *Let I be a Q–ideal of a semiring R. Then I is a k–ideal and the zero of the quotient semiring $\frac{R}{I}$. (An ideal is a k–ideal if whenever $x + i \in I$, where $x \in R$ and $i \in I$, we have $x \in I$).*

2. Quotient semiring

The following is an important role in the study of quotient semiring. We can prove easily the theorem by using mathematical induction.

THEOREM 2.1. *Let I be a Q–ideal in the semiring R. If $q_i + I \in \frac{R}{I}$ $(i = 1, \cdots, n)$, then*

$$(q_1 + I) \oplus_Q \cdots \oplus_Q (q_n + I) = q^* + I \text{ iff } q_1 + \cdots + q_n \in q^* + I$$

$$(q_1 + I) \odot_Q \cdots \odot_Q (q_n + I) = q^* + I \text{ iff } q_1 \cdots q_n \in q^* + I$$

COROLLARY 2.2 [6]. *Let I be a Q–ideal in the semiring R. For each $n \in Z^+$, we have*

$$n(q + I) = q^* + I \text{ iff } nq \in q^* + I$$

$$(q+I)^n = q^* + I \text{ iff } q^n \in q^* + I.$$

PROPOSITION 2.3. *Let I be a Q–ideal in the semiring R. If e is an idempotent in R, then a coset $q+I$ containing e is an idempotent in the quotient semiring $\frac{R}{I}$.*

Proof. Let $(q+I)^2 = q_1 + I$ for some $q_1 \in Q$. Then by Corollary 2.2 $q^2 \in q_1 + I$ and hence $q^2 = q_1 + i_1$ for some $i_1 \in I$. Since $e \in q+I, e = q+i_2$ for some $i_2 \in I$. Now, since e is an idempotent, $e = e^2 = (q+i_2)^2 = q^2 + i_3$ for some $i_3 \in I$ and hence $e = q_1 + i_1 + i_3 \in q_1 + I$. It follows that $(q_1+I) \cap (q+I) \neq \emptyset$. By the definition of Q–ideal $q_1 + I = q + I$.

LEMMA 2.4. *Let I be a Q–ideal in the semiring R. If e_i are elements of R with $e_1e_2 = 0$ and $e_i \in q_i + I$ for some $q_i \in Q$ where $i = 1, 2$, then $(q_1 + I) \odot_Q (q_2 + I) = I$.*

Proof. Let $(q_1 + I) \odot_Q (q_2 + I) = q_* + I$ for some $q_* \in Q$. Then by Theorem 2.1 we have $q_1q_2 \in q_* + I$ and hence $q_1q_2 = q_* + i_3$ for some $i_3 \in I$. Since $e_i \in q_i + I$, we have $0 = e_1e_2 = q_1q_2 + i_4$ for some $i_4 \in I$. It follows that $0 = q_* + i_3 + i_4 \in q_* + I$ and hence $(q_* + I) \cap I \neq \emptyset$. By the definition of Q–ideal we have $q_* + I = I$.

DEFINITION 2.5. *An element e of R is called idempotent of order n if there exist idempotents $e_1, \cdots, e_n$ in R with $e_ie_j = 0 \quad (i \neq j)$ such that $e = e_1 + \cdots + e_n$.*

THEOREM 2.6. *Let I be a Q–ideal in the semiring R. If e is an idempotent of order n in R, then the quotient semiring $\frac{R}{I}$ has an idempotent of order n.*

Proof. The proof is by induction on n. First, we consider the case $n = 2$. Let $e = e_1 + e_2$ where e_1, e_2 are idempotents in R with $e_1e_2 = 0$. Let $e \in q+I$, $e_i \in q_i+I$ for some $q, q_i \in Q$. Then $e = q+i_4$; $e_k = q_k+i_k$ for some i_4, $i_k \in I$ where $k = 1, 2$. Suppose $(q_1 + I) \oplus_Q (q_2 + I) = q_3 + I$ for some $q_3 \in Q$. Then $q_1 + q_2 = q_3 + i_3$ for some $i_3 \in I$. Hence we have $q + i_4 = e = q_1 + q_2 + i_1 + i_2 = q_3 + i_3 + i_1 + i_2 \in q_3 + I$. It follows from the definition of Q–ideal that $q + I = q_3 + I$. By Lemma 2.4 $(q_1 + I) \odot_Q (q_2 + I) = I$. This proves the case $n = 2$. Suppose, as the induction hypothesis, that $n > 2$ and that it holds for $n - 1$.

Let $e = e^* + e_n$ where $e^* = e_1 + \cdots + e_{n-1}$, e_i are all idempotents in R with $e_i e_j = 0$ $(i \neq j)$. Then e^* is an idempotent of order $n-1$. By induction hypothesis $q^* + I = (q_1 + I) \oplus_Q \cdots \oplus_Q (q_{n-1} + I)$ where $e^* \in q^* + I$, $e_i \in q_i + I$ for some $q^*, q_i \in Q$. Now, since $e = e^* + e_n$, we have $q + I = (q^* + I) \oplus_Q (q_n + I)$ where $e \in q + I$, $e_n \in q_n + I$ for some $q, q_n \in Q$. It follows that $q + I = (q_1 + I) \oplus_Q \cdots \oplus_Q (q_n + I)$. By Lemma 2.4 we can see $(q_i + I) \odot_Q (q_j + I) = I$ $(i \neq j)$. This completes the proof.

3. Extension of quotient halfring

We say additively cancellative semiring a *halfring*. Let R be a halfring and $R^* = \{(h,k) \mid h, k \in R\}$. In R^* define $(h,k) = (h',k')$ if and only if $h + k' = h' + k$. This gives an equivalence relation on R^*. Let $\overline{R}$ be the set of all equivalences classes in R^*. In $\overline{R}$ define $(h,k) + (h',k') = (h+h', k+k')$ and $(h,k)(h',k') = (hh' + kk', hk' + kh')$, then $\overline{R}$ is a ring with respect to these operations and R is embedded in $\overline{R}$. We identify the ordered pair (h,k) with $h - k$. Then $\overline{R} = \{h - k \mid h, k \in R\}$ is called the *ring of difference* of R and is the smallest ring containing R. If I is an ideal in a halfring R, then $\overline{I} = \{a_1 - a_2 \mid a_i \in I\}$ is an ideal in $\overline{R}$. (See Dale [5])

DEFINITION 3.1 [8]. *An ideal I of a semiring R is called semisubtractive if for $a, b \in I$, at least one of the equations $a + x = b$ or $b + x = a$ has a solution in R.*

REMARK 3.2. *Clearly every k–ideal is a semisubtractive and the solutions of the above equations belong to k–ideal.*

PROPOSITION 3.3. *If I is a Q–ideal in a halfring R, then the quotient semiring $\frac{R}{I}$ is a halfring.*

Proof. Suppose that $(q_1 + I) \oplus_Q (q_2 + I) = (q_1 + I) \oplus_Q (q_3 + I)$. Since I is a Q–ideal, I is semisubtractive and we can see that either $q_1 + q_2 = q_1 + q_3 + i_1$ or $q_1 + q_3 = q_1 + q_2 + i_2$ for some $i_1, i_2 \in I$. It follows from R is a halfring that either $q_2 = q_3 + i_1$ or $q_3 = q_2 + i_2$. This means that $q_2 + I = q_3 + I$.

THEOREM 3.4. *If I is a Q–ideal in a halfring R, then the ring of* *difference* $\overline{\frac{R}{I}}$ *of quotient halfring* $\frac{R}{I}$ *is isomorphic to quotient ring* $\frac{\overline{R}}{\overline{I}}$.

Proof. Define $\Phi : \overline{R} \longrightarrow \overline{\frac{R}{I}}$ by $(a_1 - q_2)\Phi = (q_1 + I) - (q_2 + I)$, where q_i are unique in Q such that $a_i + I \subset q_i + I$. First, we show that Φ is well-defined. Suppose $a_1 - a_2 = b_1 - b_2 \in \overline{R}$. Then $a_1 + b_2 = a_2 + b_1$. Let $(a_1 - a_2)\Phi = (q_1 + I) - (q_2 + I)$ and $(b_1 - b_2)\Phi = (q_3 + I) - (q_4 + I)$ where $a_i + I \subset q_i + I$, $b_j + I \subset q_j + I$. Let $q_* + I = (q_1 + I) \oplus_Q (q_4 + I)$ and $q^* + I = (q_2 + I) \oplus_Q (q_3 + I)$. It follows that $a_1 + b_2 = q_1 + q_4 + i_1 = q^* + i_2$ for some $i_1, i_2 \in I$ and $a_2 + b_1 = q_2 + q_3 + i_3 = q^* + i_4$ for some $i_3, i_4 \in I$. Since, $a_1 + b_2 = a_2 + b_1$, we have $q_* + I = q^* + I$ and hence $(a_1 - a_2)\Phi = (b_1 - b_2)\Phi$. Elementary calculations show that Φ is a homomorphism from $\overline{R}$ onto $\overline{\frac{R}{I}}$. Now, we can see that $\mathrm{Ker}\Phi = \overline{I}$. This completes the proof.

EXAMPLE 3.5. *Let Z_+ denote the semiring of non-negative integers with usual operarions of addition and multiplication. If $m \in Z_+$, then the ideal $(m) = \{nm \mid n \in Z_+\}$ is a Q-ideal. ([1]) We can prove that the ring of difference $\overline{Z_+/(m)}$ of quotient halfring $Z_+/(m)$ is isomorphic to quotient ring $\overline{Z}_+/\overline{(m)}$.*

Since any halfring R can be embedded in the ring of differences of R, we regard x of R as $x - 0$ and assume that $a + \overline{I} = (a - 0) + \overline{I}$ where I is an ideal of R. Then we have the following propositions. (Refer to [3])

PROPOSITION 3.6. *Let I be a Q-ideal of a halfring R. For any $q_1 \neq q_2$ in Q, we have $(q_1 + \overline{I}) \cap (q_2 + \overline{I}) = \emptyset$.*

PROPOSITION 3.7. *Let I be a Q-ideal of a halfring R. Then we have* $\frac{\overline{R}}{\overline{I}} = \{(q_1 - q_2) + \overline{I} \mid q_1, q_2 \in Q\}$.

References

1. P.J. Allen, *A fundamental theorem of homomorphisms for semirings,* Proc. A.M.S. 21 (1969) 412–416.
2. S. Bourne and H. Zassenhaus, *On a Wedderburn - Artin structure theory of a potent semiring,* Proc. N.A.S. 43 (1957) 613–615.

3. Y.B. Chun, H.S. Kim and H.B. Kim, *A study on the structure of a semiring*, Jour. N.S.R.I., Yonsei Univ. 11 (1983) 69–74.
4. L. Dale, *Standard halfrings and standard ideals*, Kyungpook Math. J. 21 (1981) 281–288.
5. L. Dale, *Extending certain semiring homomorphism to ring homomorphisms*, Kyungpook Math. J. 23 (1983) 13–18.
6. C.B. Kim, *On the quotient structure of a Euclidean semiring modulo a prime Q–ideal*, Jour. S.I., Kookmin Univ. 1 (1981) 93–102.
7. D.R. LaTorre, *A note on quotient semiring*, Proc. A.M.S. 24 (1970) 463–465.
8. H.E. Stone, *Ideals in halfrings*, Proc. A.M.S. 33 (1972) 8–14.

Chungbuk National University
Cheongju 360–763, Korea