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ALGORITHMS FOR CONSTRAINED OPTIMIZATION USING DIFFERENTIABLE PENALTY FUNCTIONS *

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0. Introduction

The constrained nonlinear programming problem is generally expressed as follows ; [minimize $f(x)$, $x \in \mathbf{R}^n$, subject to $c_i(x) = 0$, $i \in E$, $c_i(x) \geq 0$, $i \in I$]. The main purpose is to obtain the local and global solutions of this problem. In 1943, R. Courant [6] first considered a penalty function $\phi(x,\mu) = f(x) + \frac{1}{2}\mu \sum_{i \in E} [c_i(x)]^2$ and showed that constrained problems are reduced to unconstrained problems. Since the beginning of 1960's, this penalty method has made a great progress and has been studied. C.W. Caroll [2], A.V. Fiacco and G.P. McCormick [9] have researched interior penalty methods and their penalty function is $\phi(x,\mu) = f(x) + \mu \sum_{i \in I} [1/c_i(x)]$. On the other hand, exterior penalty methods have been investigated by K.Truemper [15]. P.Loridan and J.Morgan [13], and their penalty function is $\phi(x,\mu) = f(x) + \mu P(x)$ (if x is feasible, $P(x) = 0$, and if x is not feasible $P(x) > 0$).

Recently, many authors ([3], [4], [5], [10], [12]) have studied exact penalty methods which change the constrained nonlinear programming problem to a single unconstrained nonlinear programming problem. We can divide the methods largely in two in which utilize either the nondifferentiable exact penalty function or the differentiable exact penalty function. Of the two methods, we are more interested in the second methods.

We will describe in detail the results of many authors' researches. By using the nondifferentiable exact penalty function $\phi(x,\mu) = f(x)$

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$+\mu \max\{0,\ |c_i|,\ c_j\ :\ i \in E,\ j \in I\}$, D.P. Bertsekas [1] and D.G. Luenberger [14] obtained the following consequences ;

1. Let x^* be an isolated local minimum satisfying together with corresponding Lagrange multiplier vectors λ^* and μ^*, the following assumption.

Assumption ; $f,\ c_i \in C^2$ and $z^T[\nabla^2 f(x^*)$ $+\sum_{i\in E}\lambda_i^*\nabla^2 c_i(x^*) + \sum_{j\in I'}\alpha_j^*\nabla^2 c_i(x^*)]z > 0$ for all $z \neq 0$ with $\nabla c_i(x^*)z = 0 (i \in E)$ and $\nabla c_j(x^*)z = 0$ $(j \in I' = \{j \in I\ :\ c_j(x^*) = 0\})$. In addition, μ^* satisfies the strict complementarity assumption. Then, if $\mu > \sum_{i\in E}|\lambda_i^*| + \sum_{j\in I}|\alpha_j^*|$, x^* is an isolated unconstrained local minimum of $\phi(x,\mu)$.

2. Let $X \subseteq \mathbf{R}^n$ be a compact set such that, for all $x \in X$, the set of gradients $\{\nabla c_i(x)\}$ is linearly independent. Then, there exists a $\mu^* > 0$ such that for every $\mu > \mu^*$, (a) If x^* is a critical point of $\phi(x,\mu)$ and $x^* \in X$, there exist α^*, λ^* such that $(x^*,\ \alpha^*,\ \lambda^*)$ is a Kuhn–Tucker point. (b) If $(x^*,\ \alpha^*,\ \lambda^*)$ is a Kuhn–Tucker point and $x^* \in X$, then x^* is a critical point of $\phi(x,\mu)$.

M.R.Hestness [11] considered the penalty function(for $I = \phi$). $\phi(x,\lambda,\mu) = f(x) + \sum_{i\in E}\lambda_i c_i(x) + \frac{1}{2}\mu\sum_{i\in E}[c_i(x)]^2$ and obtained solutions by unconstrained minimization problems. In 1979, G.Dipillo and L.Grippo [7] pointed out the defects on the Hestness' method and proposed their own exact method to utilize the differentiable penalty function. For $I = \phi$, their penalty function is $\phi(x,\lambda,\mu) = \lambda^T c(x) + \mu\|c(x)\|^2 + \|M(x)(\nabla f(x) + \nabla c(x)\lambda)\|^2$ and their result is as follows ;

1. Let $(x^*,\ \lambda^*)$ be a critical point of $L(x,\lambda)$. Assume that $M(x^*)$. $[\partial c(x^*)/\partial x]^T$ has full row rank and $x^T\nabla_x^2 L(x^*,\ \lambda^*)x > 0$ for all x, with $x \neq 0$ and $[\partial c(x^*)/\partial x]x = 0$ where $L(x,\lambda) = f(x) + \lambda^T c(x)$. If x^* is a local minimum, then there exists a $\mu^* > 0$ such that for all $\mu \geq \mu^*$, $(x^*,\ \lambda^*)$ is an isolated local minimum of $\phi(x,\lambda,\mu)$

2. Let $X \times \Lambda$ be a compact subset of $X^* \times \mathbf{R}^n$ and assume that $M(x)\nabla c(x)$ is nonsingular for all $x \in X$. Then there exists a $\mu^* > 0$ such that for all $\mu \geq \mu^*$, if $(x^*,\ \lambda^*)$ is an unconstrained local minimum of $\phi(x,\lambda,\mu)$ belonging to $X \times \Lambda$, then x^* is a local minimum.

G.Dipillo, L.Grippo [8], C.Vinante, and S.Pintos [16] extended those results to the case that $I \neq \phi$ and $E = \phi$.

C.Vinante and S.Pintos utilize the differentiable exact penalty function $\phi(x, \lambda, \mu, \alpha) = f(x) + \lambda^T c(x) + \mu\|\nabla f(x) + \nabla c(x)\lambda\|^2 - \mu\|d\|^2$, where $d_j = -\min[0, c_j(x) + (\frac{1}{\mu}\lambda_j/2, \ (1 + 4d\lambda_j))]$.

In this paper, we obtain a new penalty function by which the constrained minimization problem is converted into the unconstrained minimization problem. From this penalty function, we generate a differentiable penalty function which is applicable to practical problems, and show that the constrained problem is equivalent to the unconstrained problem under certain assumptions. On the basis of this equivalence, we make our penalty method.

1. Preliminaries

We consider the constrained problem ;

[1.1] minimize $f(x)$, subject to $x \in S = \{x : g(x) = 0\}$, where f is a continuous function from $\mathbf{R}^n$ to $\mathbf{R}$ and g is a continuous function from $\mathbf{R}^n$ to $\mathbf{R}$. And, we introduce well-known optimality conditions which we are going to make use of in section 2 and section 3.

THEOREM 1.1. *(first-order necessary conditions)*
Let $f \in C^1$ and $g \in C^1$. Suppose that $\overline{x}$ is a local minimum for [1.1]. Then there is a $\lambda \in \mathbf{R}^n$ such that $f'(\overline{x}) - \lambda^T g'(x) = 0$.

THEOREM 1.2. *(second-order sufficient conditions)*
Let $f \in C^2$ and $g \in C^2$. Let $\overline{x}$ be a point feasible to the constraints of [1.1]. Suppose that the first-order necessary conditions are satisfied at $\overline{x}$ and that $z^T[f''(\overline{x}) - \sum_{j=1}^{m} \lambda_j g_j''(\overline{x})]z > 0$ for all z, where $g'(\overline{x})z = 0$. Then, $\overline{x}$ is an isolated local minimum for [1.1].

In the sequel, the column of the matrix P will be denoted by P^i, and the jth row of P will be denoted by P_j.

2. The inequality–constrained problem

We consider the inequality–constrained nonlinear programming problem ;

[2.1] minimize $f(x)$, subject to $x \in S = \{x : g(x) \leq 0\}$, where f is a continuous function from $\mathbf{R}^n$ to $\mathbf{R}$ and g is a continuous function from $\mathbf{R}^n$ to $\mathbf{R}$, and assume that $0 \in$ int S.

For each $x \in \mathbf{R}^n$, we let λ^* be the optimal solution of the maximization problem;

[maximize $f(x)$, subject to $\lambda x \in S$, $0 \leq \lambda \leq 1$], and $h(x) = \lambda^* x$. We put $d(x) = x - h(x)$ and $P(x) = f[x - d(x)] + q\|d(x)\|^2$ $(q > 0)$. $P(x)$ is our penalty function.

THEOREM 2.1. *If $\overline{x}$ is a local unconstrained minimum for $P(x)$, then it is a local minimum for [2.1].*

Proof. We assume that $d(x) \neq 0$ and $h(x) = \lambda^* x$. For ε with $0 < \varepsilon \leq 1$, $x - \varepsilon d(x) = [\varepsilon\lambda^* + (1-\varepsilon)]x$. By the definition of h, $h[x - \varepsilon d(x)] = \lambda^* x$. $d[x - \varepsilon d(x)] = x - \varepsilon d(x) - h[x - \varepsilon d(x)] = (1-\varepsilon)(1-\lambda^*)x$.

$$
\begin{aligned}
P[x - \varepsilon d(x)] &= f\{x - \varepsilon d(x) - d[x - \varepsilon d(x)]\} + q\|d[x - \varepsilon d(x)]\|^2 \\
&= f(\lambda^* x) + q\|(1-\varepsilon)(1-\lambda^*)x\|^2 \\
&= f[x - d(x)] + (1-\varepsilon)q\|(1-\lambda^*)x\|^2 \\
&< f[x - d(x)] + q\|(1-\lambda^*)x\|^2 \\
&= f[x - d(x)] + q\|d(x)\|^2 \\
&= P(x).
\end{aligned}
$$

Thus, $P[x - \varepsilon d(x)] < P(x)$. If $d(\overline{x}) \neq 0$, $P[\overline{x} - \varepsilon d(\overline{x})] < P(\overline{x})$ for ε with $0 < \varepsilon \leq 1$. This contradicts to the assumption. Hence, $d(\overline{x}) = 0$ and $f(\overline{x}) = P(\overline{x})$. By the assumption, there is a neighborhood $N(\overline{x}\,;\,\delta)$ such that $P(x) \geq P(\overline{x})$ for all $x \in N(\overline{x}\,;\,\delta)$. Let $x \in S \cap N(\overline{x}\,;\,\delta)$. Then, $f(x) = P(x) \geq P(\overline{x}) = f(\overline{x})$. Hence, the above theorem holds.

THEOREM 2.2. *Suppose that h is continuous. If $\overline{x}$ is a local minimum for [2.1], then it is an unconstrained local minimum for $P(x)$*

Proof. If the theorem is not true, there is an infinite sequence of points

$\{x_k\}$ such that $x_k \to \overline{x}$ and $P(x_k) < P(\overline{x})$.

$$\begin{aligned} f[x_k - d(x_k)] &< f[x_k - d(x_k)] + qd(x_k)^T.d(x_k) \\ &= P(x_k) \\ &< P(\overline{x}) \\ &= f(\overline{x}) \end{aligned}$$

Hence $f[x_k - d(x_k)] < f(\overline{x})$. By the continuity of $h, h(x_k) \to h(\overline{x}) = \overline{x}$. Hence, $x_k - d(x_k) = h(x_k) \to \overline{x}$. This contradicts the assumption that $\overline{x}$ is a local minimum for [2.1]. Hence, the above theorem holds.

3. The equality–contrained problem

We consider the equality–constrained nonlinear problem ; [3.1] minimize $f(x)$, subject to $x \in S = \{x : g(x) = 0\}$, where f is a continuous function form $\mathbf{R}^n$ to $\mathbf{R}$ and g is a continuous function form $\mathbf{R}^n$ to $\mathbf{R}$. Let $h(x)$ be the optimal solution of the minimization problem ; [3.2] minimize $\|x - z\|^2$, subject to $g(z) = 0$. Usually the vector $h(x)$ is unique, but if it is not, to complete the definition of h, the following is used ; Define $H(x) = \{h : x - h$ solves [3.2] $\}$. Let $h(x)$ be a vector from $H(x)$ such that $f[h(x)]$ is minimal. And we let $d(x) = x - h(x)$ and $P(x) = f[x - d(x)] + q\|d(x)\|^2$, $(q > 0)$. $P(x)$ is our penalty function.

THEOREM 3.1. *If $\overline{x}$ is a local unconstrained minimum for $P(x)$, it is a local minimum for [3.1].*

Proof. By the same method of Theorem 2.1, we can prove the above result.

THEOREM 3.2. *Suppose that $h(x)$ is unique and continuous. If $\overline{x}$ is a local minimum for [3.1], it is a local unconstrained minimum for $P(x)$.*

Proof. By the same method of Theorem 2.2, we can prove the above result.

THEOREM 3.3. *Suppose that $f \in C^3$ and $g \in C^3$. Suppose that $\overline{x}$ is a point such that $g'(\overline{x})$ has full row rank, and that $h(x)$ is unique and continuously differentiable in a neighborhood of $\overline{x}$. If $\overline{x}$ is an isolated unconstrained minimum for $P(x)$, that is, if $P'(x) = 0$ and $P''(\overline{x})$ is* *a positive definite matrix, then $\overline{x}$ satisfies the second–order sufficient conditions for an isolated local minimum for [3.1].*

Proof. $P'(x) = f'[x - d(x)][I - d'(x)] + 2qd(x)^T d'(x)$.

$$P''(x) = \sum_{i=1}^{n} \frac{\partial f[x - d(x)]}{\partial[x_i - d_i(x)]} \cdot [-d_i''(x)] + [I - d'(x)^T] f''[x - d(x)][I - d'(x)] + 2qd'(x)^T d'(x) + 2q \sum_{i=1}^{n} d_i(x) d_i''(x).$$

Consider any point x near $\overline{x}$. Clearly $h(x)$ is close to $\overline{x}$ and the matrix $g'[h(x)]$ has rank m. By the first–order necessary conditions for [3.2], there exists a $\lambda \in R^n$ such that $2[x - h(x)] - g'[h(x)]^T = 0$.

$$g'[h(x)]^T \lambda = -2[h(x) - x].$$

$$\{g'(h(x)]g'[h(x)]^T\}^{-1} g'[h(x)]g'[h(x)]^T \lambda = -2\{g'[h(x)]g'[h(x)]^T\}^{-1} g'[h(x)][h(x) - x].$$

Hence, $\lambda = -2\{g'[h(x)]g'[h(x)]^T\}^{-1} g'[h(x)][h(x) - x]$.
Therefore, $d(x) - g'[h(x)]^T \{g'[h(x)]g'[h(x)]^T\}^{-1} g'[h(x)] = 0$.
Let $G(x) = I - g'[h(x)]^T \{g'[h(x)]g'[h(x)]^T\}^{-1} g'[h(x)]$.
Then,

$$(1) \qquad G(x)d(x) = 0,$$

By differentiating (1), we have

$$(2) \qquad G(x)d'(x) + \sum_{i=1}^{n} d_i(x)[G^i(x)]' = 0$$

$$(3) \qquad g[x - d(x)] = g[h(x)] = 0.$$

By differentiating (3), we have

$$\begin{bmatrix} g_1'[x-d(x)][I-d'(x)] \\ \vdots \\ g_m'[x-d(x)][I-d'(x)] \end{bmatrix} = 0 \tag{4}$$

When $d(x) = 0$, (3) implies that $g'(x) - g'(x)d'(x) = 0$.
Hence

$$g'(x) = g'(x)d'(x). \tag{5}$$

From (2) and (5),

$$\begin{aligned} 0 &= G(x)d(x) \\ &= d'(x) - g'[h(x)]^T\{g'[h(x)]g'[h(x)]^T\}^{-1}g'[h(x)]d'(x) \\ &= d'(x) - g'(x)^T[g'(x)g'(x)^T]^{-1}g'(x)d'(x) \\ &= d'(x) - g'(x)^T[g'(x)g'(x)^T]^{-1}g'(x). \end{aligned}$$

Hence

$$d'(x) = g'(x)^T[g'(x)g'(x)^T]^{-1}g'(x). \tag{6}$$

By differentiating (4), we have

$$\sum_{i=1}^{n} \frac{\partial g_i[h(x)]}{\partial h_i(x)}[-d_i(x)]'' + [I - d'(x)^T]g_j''[h(x)][I - d'(x)] = 0. \tag{7}$$

Because $d(\overline{x}) = 0$, the formula (6) can be used.

$$\begin{aligned} 0 &= P'(\overline{x}) \\ &= f'(\overline{x})[I - d'(\overline{x})] + 2qd(\overline{x})^T d'(\overline{x}) \\ &= f'(\overline{x})[I - d'(\overline{x})] \\ &= f'(\overline{x}) - f'(\overline{x})d'(\overline{x}) \\ &= f'(\overline{x}) - f'(\overline{x})g(\overline{x})^T[g'(\overline{x})g'(\overline{x})^T]^{-1}g'(\overline{x}). \end{aligned}$$

Let $u(\overline{x})^T = f'(\overline{x})g'(\overline{x})^T[g'(\overline{x})g'(\overline{x})^T]^{-1}$.

Then,

$$f'(\overline{x}) - u(\overline{x})^T g'(\overline{x}) = 0. \tag{8}$$

Hence, the first–order necessary conditions are satisfied at $\overline{x}$ for a constrained minimum.

From (7) and (8),

$$\sum_{i=1}^{n} \frac{\partial f(\overline{x})}{\partial x_i}[-d_i(\overline{x})]'' = \sum_{j=1}^{m} u_j(\overline{x})\{\sum_{i=1}^{n} \frac{\partial g_i(\overline{x})}{\partial x_i}[-d_i(x)]''\}$$
$$= -[I - d'(\overline{x})^T][\sum_{j=1}^{m} u_j(\overline{x})g_j''(\overline{x})][I - d'(\overline{x})]$$

Hence,

$$P''(\overline{x})$$
$$= -[I - d'(\overline{x})^T][\sum_{j=1}^{m} u_j(\overline{x})g_j''(\overline{x})][I - d'(\overline{x})] + [I - d'(\overline{x})^T]f''(\overline{x})[I - d'(\overline{x})]$$
$$+ 2qg'(\overline{x})^T[g'(\overline{x})g'(\overline{x})^T]^{-1}g'(\overline{x})g'(\overline{x})^T[g'(\overline{x})g'(\overline{x})^T]$$
$$= [I - d'(\overline{x})^T][f''(x) - \sum_{j=1}^{m} u_j(\overline{x})g_j''(\overline{x})][I - d'(\overline{x})]$$
$$+ 2qg'(\overline{x})[g'(\overline{x})g'(\overline{x})^T]^{-1}g'(\overline{x})$$
$$= G(\overline{x})[f''(\overline{x}) - \sum_{j=1}^{m} u_j(\overline{x})g_j''(\overline{x})]G(\overline{x}) + 2qg'(\overline{x})^T[g'(x)g'(\overline{x})^T]^{-1}g'(\overline{x}).$$

By the positive definiteness of $P''(\overline{x})$,

$$z^T G(\overline{x})[f''(\overline{x}) - \sum_{j=1}^{m} u_j(\overline{x})g_j''(\overline{x})]G(\overline{x})z$$
$$= z^T[f''(\overline{x}) - \sum_{j=1}^{m} u_j(\overline{x})g_j''(\overline{x})]z > 0$$

for all z where $g'(\overline{x})z = 0$.

Thus the second–order sufficient conditions are satisfied.

We consider the following penalty function $M(x)$ using approximation and assume that $g'(x)$ has full row rank.

$$\begin{aligned} M(x) =& f(x) - f'(x)g'(x)^T[g'(x)g'(x)^T]^{-1}g(x) \\ & - \frac{1}{2}f'(x)g'(x)^T[g'(x)g'(x)^T]^{-1} \\ & \begin{bmatrix} g(x)^T[g'(x)g'(x)^T]^{-1}g'(x)g_1''(x)g'(x)^T[g'(x)g'(x)^T]^{-1}g(x) \\ \vdots \\ g(x)^T[g'(x)g'(x)^T]^{-1}g'(x)g_m''(x)g'(x)^T[g'(x)g'(x)^T]^{-1}g(x) \end{bmatrix} \\ & + \frac{1}{2}g(x)^T[g'(x)g'(x)^T]^{-1}g'(x)f''(x)g'(x)^T[g'(x)g'(x)^T]^{-1}g(x) \\ & + qg(x)[g'(x)g'(x)^T]^{-1}g'(x)[g'(x)g'(x)^T]^{-1}g(x) \end{aligned}$$

THEOREM 3.4. *Suppose that $f \in C^2$ and $g \in C^2$. Let $\overline{x}$ be an unconstrained local minimum for $M(x)$. If $g(\overline{x}) = 0$, then $\overline{x}$ is a constrained local minimum for [3.1].*

Proof. Let $x \in S$ be any point close to $\overline{x}$.
Since $h(x) = 0$, $f(x) = M(x)$.
Since $\overline{x}$ is a local minimum for $M(x)$, $M(x) \geq M(\overline{x}) = f(\overline{x})$.
Hence $f(x) \geq f(\overline{x})$.

THEOREM 3.5. *Suppose that $f \in C^3$ and $g \in C^3$. Suppose that $\overline{x}$ is a point where $g'(\overline{x})$ has full row rank, and that $\overline{x}$ satisfies the second–order sufficiency conditions for an isolated local minimum. Then $\overline{x}$ is an isolated unconstrained local minimum for $M(x)$ for any value of $q > 0$.*

Proof. Because $\overline{x}$ is feasible,

$$g(\overline{x}) = 0 \tag{1}$$

By the first–order necessary condition,

$$f'(\overline{x}) - f'(\overline{x})g'(\overline{x})[g'(\overline{x})g'(\overline{x})^T]^{-1}g'(\overline{x}) = 0 \tag{2}$$

Hence $M'(\overline{x}) = 0$.

From (1) and (2),

$$M''(\overline{x}) = \{I - g'(\overline{x})^T\ [g'(\overline{x})g'(\overline{x})^T]^{-1}g'(\overline{x})\} \cdot [f''(\overline{x}) - \textstyle\sum_{i=1}^m u_i(\overline{x})g_i''(\overline{x})]$$
$$\cdot \{I - g'(\overline{x})^T\ [g'(\overline{x})g'(\overline{x})^T]^{-1}g'(\overline{x})\} + 2qg'(\overline{x})^T[g'(\overline{x})g'(\overline{x})^T]^{-1}g'(\overline{x}).$$

The second–order sufficient conditions imply that $M''(\overline{x})$ is positive definite for every $q > 0$.

Hence $\overline{x}$ is an isolated unconstrained local minimum for $M(x)$ for any $q > 0$.

By using $M(x)$, we obtain local minimizers for [3.1].

To get local minimizers for $M(x)$, we make use of algorithms which are generated by various methods.

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