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SEMIPRIME RINGS SATISFYING CERTAIN ANNIHILATOR CONDITIONS

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Throughout this note R denotes a ring with identity. In [2], R.Y.C. Ming studied left subset annihilator rings (l.s.a.r. for brevity). A l.s.a.r. is defined as a ring in which every essential left ideal of R is the left annihilator of some subset of R. A r.s.a.r. is defined similarly. He has proved [2, Theorem 2] : If R is a semiprime l.s.a.r. then R is semisimple Artinian.

In this paper, annihilator conditions for rings to be semisimple Artinian are considered.

We obtain some characterizations of a semisimple Artinian ring and the Ming's result in [2, Theorem 2]. Moreover, a theorem of Ming's [3, Theorem 3] will be improved.

For definitions and notations we refer the reader to [1].

We start with the following lemma.

LEMMA 1. *Let R be a ring. The following conditions are equivalent.*

(1) *R is semisimple Artinian.*
(2) *Every maximal right ideal of R is a direct summand.*
(3) *R has no essential maximal right ideals.*

The proof is not necessary because it is routine.

THEOREM 2. *Let R be a ring. The following conditions are equivalent.*

(1) *R is semiprime and $l(M) \neq 0$ for every maximal right ideal M of R, where $l(M)$ is the left annihilator of M.*
(2) *R is semisimple Artinian.*

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Proof. (1) $\Longrightarrow$ (2) : Assume that R has an essential maximal right ideal M. Let x, y be any nonzero elements of $l(M)$. Then $M \cap yR \neq 0$. Thus we can choose r in R such that $x(yr) = 0$ and $yr \neq 0$. Now $xy \in l(M)$, this shows $\mathcal{R}(xy) \supseteq M$, where $\mathcal{R}(xy)$ is the right annihilator of xy. Since M is a maximal right ideal, either $\mathcal{R}(xy) = R$ or $\mathcal{R}(xy) = M$. If $\mathcal{R}(xy) = R$, then $xy = 0$. It implies $l(M)^2 = 0$, or equivalently $l(M) = 0$ because R is semiprime. So it is contradict to our assumption on R. Next we suppose that $\mathcal{R}(xy) = M$. Then $(xy)r = x(yr) = 0$ implies $r \in \mathcal{R}(xy) = M$. Thus $yr = 0$, since $y \in l(M)$. But we have already $yr \neq 0$, thus it is absurd. Therefore R has no essential maximal right ideals. It follows from Lemma 1 that R is semisimple Artinian.

(2) $\Longrightarrow$ (1) : By lemma 1, $M = eR$ for any maximal right ideal M of R and $e = e^x$ in R. This $l(M) = l(eR) = R(1 - e) \neq 0$. Since R is semisimple Artinian, R is always semiprime

It is well known [For example, 1, p.235–236] that a ring R is right Kasch iff $l(A) \neq 0$ for every right ideal $A \neq R$.

COROLLARY 3. *A semiprime right Kasch ring is semisimple Artinian.*

Proof. Obvious.

THEOREM 4. [Ming 2, Theorem 2]. *If R is a semiprime* r.s.a.r. *(resp.* l.s.a.r.*), then R is semisimple Artinian.*

Proof. Suppose R is not semisimple Artinian. Then there exists a essential maximal right ideal M of R. Since R is r.s.a.r., $l(M) \neq 0$. But the proof of theorem 2 reveals a contradiction.

The next theorem is, in a sense, a generalization of a theorem of Ming's [3, Theorem 3].

THEOREM 5. *Let M be any maximal right ideal in R. The following conditions are equivalent.*

(1) *R is semisimple Artinian.*
(2) *$l(M)$ contains a nonzero idempotent element.*
(3) *$l(M)$ is not nil.*

Proof. (1) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (3) are obvious. (3) $\Longrightarrow$ (1) : Since $l(M)$ is not nil we can choose an element a in $l(M)$ which is not nilpotent. Thus we may assume $a^2 \neq 0$. Now $M \subseteq \mathcal{R}(a) \subseteq \mathcal{R}(a^2)$, hence $M = \mathcal{R}(a) = \mathcal{R}(a^2)$. Suppose that M is essential in R. Then $M \cap aR \neq 0$. Thus there exists $r \in R$ such that $ar \in M$ and $ar \neq 0$. But $0 = a(ar) = a^2 r$, this shows $r \in \mathcal{R}(a^2) = M$. Hence $ar = 0$, which is absurd. Therefore M can't be essential in R. By Lemma 1, R is semisimple Artinian.

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