HOLOMORPHIC VECTOR BUNDLES OVER COMPLEX TORI

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1. Introduction

The purpose of this paper is to study the holomorphic vector bundles over a complex torus. The theory of vector bundles of rank r over a g-dimensional complex torus is not sufficiently developed except for $r=1$ and $g=1$. In his paper [1], Atiyah classified the vector bundles over an elliptic curve. Many algebraic geometers, e. g., Weil, Mumford, studied the line bundles over a complex torus. The theory of line bundles is the theory of theta functions. Indeed, the classification of vector bundles over a complex torus $T=V/L$ corresponds to that of automorphic factors for L, where V is a g-dimensional complex vector space and L is a lattice in V. In general, it is a very difficult problem to classify the automorphic factors for a discrete group. Matsushima [1] and Morimoto [1] classified the flat vector bundles over a complex torus. It is equivalent to the problem of the classification of representations of a lattice group L. And Matsushima [2] and Hano [1] characterized the projectively flat vector bundles over a complex torus. Mukai [1] introduced the concept of semi-homogeneous vector bundles over an abelian variety and characterized the semi-homogeneous vector bundles. In fact, semi-homogeneous vector bundles corresponds to projectively flat vector bundles. In this paper, we characterize the projectively flat vector bundles over a complex torus completely and investigate the connection between those vector bundles and the Heisenberg group. I would like to remark that it is so interesting to classify the automorphic factors corresponding to the stable vector bundles.

In Section 2, we review the basic properties of complex tori. We will omit the proofs and the details. In Section 3, we describe the automorphic factors for the holomorphic vector bundles and in particular, we write the automorphic factors for line bundles explicitly. In Section 4, we review the general results about line bundles over a complex torus we will use in the following sections. For details, we refer to Mumford [1] and Yang [1]. In Section 5, we characterize the projectively flat vector bundles over a complex torus completely and we discuss the stability of those vector bundles. In Section 6, we investigate the connection between the Heisenberg groups and the projectively flat vector bundles.

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2. Complex tori

In this section, we briefly review the theory of complex tori. Proofs will be omitted.

Let $T = V/L$ be a complex torus of dimension g, where V is a g-dimensional comlex vector space and L is a lattice of rank $2g$ in V. Then T is a connected, compact commutative complex Lie group. A complex torus T is called an *abelian variety* if it is a projective variety, i. e., it can be holomorphically embedded in a complex projective space.

DEFINITION 1.1. A morphism $\phi : T_1 \rightarrow T_2$ of complex tori T_1, T_2 is said to be an isogeny if it is a surjective homomorphism with finite kernel. The order of the kernel is called the degree of ϕ. We say that two complex tori T_1 and T_2 are isogeneous, denoted by $T_1 \sim T_2$ if there exists an isogeny between T_1 and T_2.

If $\phi_1 : T_1 \sim T_2$ and $\phi_2 : T_2 \rightarrow T_3$ are isogenies, then so is $\phi_2 \circ \phi_1$ and the degrees multiply : $\deg(\phi_2 \circ \phi_1) = \deg \phi_2 \deg \phi_1$. The following proposition shows that $\sim$ is an equivalence relation.

PPROPOSITION 1.2. *Let T_1 and T_2 be two complex tori of the same dimension g. If $\phi ; T_1 \rightarrow T_2$ is an isogeny of degree m, then there*

exists a unique isogeny $\psi : T_2 \to T_1$ of degree m^{2g-1} such that $\psi \circ \phi = mI_1$ and $\phi \circ \psi = mI_2$, where I_1 (resp. I_2) denotes the identity of T_1 (resp. T_2). ψ is called the dual isogeny to ϕ.

Let $T = V/L$ be a g-dimensional complex torus. A Hermitian from H on V is called a *Riemann form* for the complex torus $T = V/L$ if

(i) H is nondegenerate,

(ii) *Im* $H = E$ is integral valued on the lattice L.

The following theorem is well known.

THEOREM A (Mumford[1], p. 35). *Let $T = V/L$ be a g-dimensional complex torus. Then the following are equivalent.*

(1) *T is an abelian variety,*

(2) *there exist g algebraically independent meromorphic functions on T,*

(3) *there exists a positive definite Riemann form H on V.*

EXAMPLE. Let ω be an elements in the upper-half plane. Now we consider the lattice $L = \{n + m\omega \mid n, m \in Z\}$ in C. Then $T = C/L$ is a one dimensional complex torus. We define a Hermitian form H on C by

$$H(z, w) = \frac{z \cdot w}{Im \, \omega} \text{ where } z, \ w \in C.$$

Then H is clearly a positive definite Riemann form on C. By Theorem A, there is a projective embedding of T in a complex projective space. In fact, several projective embeddings of T are well-known in the classical theory: for example, the Weierstrass φ-function

$$\varphi(z) = \frac{1}{z^2} + \sum_{(n,m) \neq (0,0)} \left[\frac{1}{(z - n - m\omega)^2} - \frac{1}{(n + m\omega)^2} \right]$$

is a meromorphic function, periodid with respect to 1, ω, with poles at the points $n + m\omega \in L$. The map

$$z \longrightarrow (1, \varphi(z), \varphi'(z))$$

induces an ismorphism of T with a plane cubic curve of the from $X_0 X_2^2 = 4X_1^3 + aX_0^2 X_1 + bX_0^3$ for suitable constants a, b depending on ω. We remark that the Weierstrass φ-function satisfies the following nonlinear differential equation

$$[\varphi'(z)]^2 = 4\varphi^3(z) - 60G_4\varphi(z) - 140G_6,$$

where $G_k (k \geq 3)$ is the Eisenstein series of order k. Here, in fact, $a = -60G_4$ and $b = -140G_6$.

Restricting the Riemann from shows that a subtorus of an abelian variety is again an abelian variety. One can show that a quotient of an abelian variety is also an abelian variety. This is a consequence of the following theorem.

THEOREM B (Poincaré Reducibility Theorem). *Suppose A is an abelian variety and $A_1 \subset A$ an abelian subvariety. Then there exists an abelian subvariety A_2 such that $A_1 \cap A_2$ is finite and A is isogeneous to $A_1 \times A_2$.*

Let $T = V/L$ be a complex torus of dimension g. Let $T_x : T \rightarrow T$ be the translation by $x \in T$. We have a Kaehler metric on T which is invariant under the translations. Since T_x preserves a Kaehler metric on T, T_x^* sends harmonic forms into harmonic forms. Since T_x is homotopic to the identity map, the map
$$T_x^* : H^k(T) \rightarrow H^k(T)$$
is the identity map, where $H^k(T)$ denotes the space of all harmonic k-forms on T. If $x \in T$, $T_{x,c}(T) = T_x'(T) + \overline{T_x'(T)}$. We identify $T_x'(T)$ with V. An element $\theta \in \Lambda^k(T_x'(T))^* \cong \Lambda^k V^*$ extends a holomorphic, translation-invariant k-forms ω_θ on T. Indeed, we define $(\omega_\theta)_y = T_{x-y}^*(\theta)$ for any $y \in T$. Then the map defines a sheaf homomorphism
$$\mathcal{O}_T \otimes_C \Lambda^k V^* \rightarrow \Omega^k$$
which is actually a sheaf homomorphism, where Ω^k is the sheaf of holomorphic k-forms on T and $\mathcal{O}_T$ is the structure sheaf on T, simply denoted by $\mathcal{O}$. Since $T \cong (S^1)^{2g}$ topologically, $\dim_C H^k(T) = \binom{2g}{k}$. Let $I^k(T)$ be the space of all translation-invariant k-forms on T. Then $\dim I^k(T) = \binom{2g}{k}$ and hence $H^k(T) = I^k(T)$. In fact,
$$H^k(T) = I^k(T) = \Lambda^k(V^* \oplus \overline{V}^*)$$
$$= \oplus_{p+q=k}(\Lambda^p V^* \otimes \Lambda^q \overline{V}^*).$$

THEOREM C. $H^q(T, \mathcal{O}) \cong \Lambda^q \overline{V}^*$ *for all q.*

For the proof of Theorem C, see Mumford ([1], p. 4).

$$H^q(T, \Omega^p) \cong H^q(T, \mathcal{O} \otimes \Lambda^p V^*)$$
$$\cong H^q(T, \mathcal{O}) \otimes_c \Lambda^p V^*$$
$$\cong \Lambda^q \bar{V}^* \otimes_c \Lambda^p V^* \text{ by Theorem } C.$$

Thus we have

$$H^k(T, C) \cong H^k(T) \text{ by the Hodge theorem}$$
$$\cong \bigoplus_{p+q=k} (\Lambda^p V^* \otimes \Lambda^q \bar{V}^*)$$
$$\cong \bigoplus_{p+q=k} H^q(T, \Omega^p).$$

This is a so-called famous Hodge theorem.

REMARKS. (1) By Theorem C, the natural map induced by cup product

$$\Lambda^q(H^1(T, \mathcal{O})) \to H^q(T, Q)$$

is an isomorphism.

(2) We consider the three sheaves on T, embedded in one another as follows:

$$Z \subset C \subset \mathcal{O},$$

where Z and C are the constant sheaves on T. Then we have the following:

$$\begin{array}{ccccc}
H^1(T, Z) & \xrightarrow{\alpha} & H^1(T, C) & \xrightarrow{\beta} & H^1(T, \mathcal{O}) \\
\| \wr & & \| \wr & & \| \wr \\
\mathrm{Hom}(L, Z) & & V^* \oplus \bar{V}^* & & \bar{V}^* \\
\| & & \| & & \\
L^* & & \mathrm{Hom}_R(V, C) & &
\end{array}$$

Let A^k(resp. $A^{k,0}$, $A^{0,k}$) be the sheaf of C^∞ k-forms(resp. of type $(k, 0)$, of type $(0, k)$) on T. Let $A_{0,1} : A^1 = A^{1,0} \oplus A^{0,1} \to A^{0,1}$ be the projection. Then we have the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & C & \longrightarrow & A^0 & \xrightarrow{d} & A^1 & \longrightarrow 0 \\
 & & \| & & \| & & \downarrow A_{0,1} & \\
0 & \longrightarrow & \mathcal{O} & \longrightarrow & A^{0,0} & \xrightarrow{\partial} & A^{0,1} & \longrightarrow 0.
\end{array}$$

Hence we have

$$\begin{array}{ccccc}
H^1(T) = V^* \oplus \bar{V}^* & \longrightarrow & H^0(T, A^1) & \longrightarrow & H^1(T, C) \\
\downarrow \text{proj} & & \downarrow A^{0,1} & & \downarrow \beta \\
\bar{V}^* & \longrightarrow & H^0(T, A^{0,1}) & \longrightarrow & H^1(T, \mathcal{O}).
\end{array}$$

Thus β is surjective.

3. Automorphic factors

Let V be a g-dimensional complex vectr space and L a lattice in V. Then V is the universal covering space of the complex torus $T=V/L$. Clearly the projection $\pi : V \to T$ is holomorphic and L is the fudamental group of T.

DEFINITION 3.1. An *automorphic factor* of rank r with respect to the lattice L in V is a holomorphic mapping $J : L \times V \to GL(r;C)$ satisfying the condition

$$J(\alpha+\beta, z) = J(\alpha, \beta+z)J(\beta, z) \text{ for all } \alpha, \beta \in L, \ z \in V.$$

Two automorphic factors J and $\tilde{J}$ of the same rank r with respect to a lattice L are said to be holomorphically equivalent (simply equivalent) if there exists a holomorphic mapping $h : V \to GL(r;C)$ such that

$$\tilde{J}(\alpha, z) = h(z+\alpha)J(\alpha, z)h(z)^{-1} \text{ for each } \alpha \in L, \ z \in V.$$

An automorphic factor $J : L \times V \to GL(r, C)$ is said to be *flat* if the mapping J is costant on V. Thus a flat automorphic factor consists of an element $J \in \mathrm{Hom}(L, GL(r;C))$ of the set of all group homomorphisms from L into $GL(r;C)$.

Let E be a holomorphic vector bundle of rank r over a complex torus $T=V/L$. If $\pi : V \to T$ is the projection, then $\pi^* E = \tilde{E}$ is holomorphically trivial since V is a contractible Stein manifold. It is Grauert's result that any topologically trivial holomorphic vector bundle over a Stein manifold is holomorphically trivial (Grauert [1]). Having fixed the isomorphism $\tilde{E} = V \times C^r$, each bundle homomorphism $\tilde{\alpha} : \tilde{E} \to \tilde{E}$ induced by a covering transformation $\alpha : V \to V$ ($\alpha \in L$) must be of the form

$$\tilde{\alpha}(z, \xi) = (z+\alpha, J(\alpha, z)\xi), \ \alpha \in L, \ z \in V, \ \xi \in C^r,$$

where $J : L \times V \to GL(r;C)$ is an automorphic factor of rank r for L. The mapping J is called the automorphic factor for the bundle E. Conversely, given an automorphic factor $J : L \times V \to GL(r:C)$, we may regard L as a group of biholomorphic mappings from $V \times C^r$ to itself by setting $\alpha(z, \xi) = (z+\alpha, J(\alpha, z)\xi)$ for $\alpha \in L$. Then the quotient $E = V \times C^r/L$ is a holomorphic vector bundle over a complex torus $T = V/L$ such that $\pi^* E = V \times C^r$.

Let E and E_1 be holomorphic vector bundles over a complex torus T of rank r and rank s respectively. Any bundle homorphism $\pi : E \to E_1$ induces a bundle homomorphism $\pi : V \times C^r \to V \times C^s$ which commutes with the action of L. Thus π must be of the form
$$\pi(z, \xi) = (z, h(z)\xi), \quad z \in V, \quad \xi \in C^r,$$
where $h : V \to M_{s,r}$ is a holomorphic mapping from V into the space $M_{s,r}$ of all $s \times r$ complex matrices. Moreover, if J and J_1 are the automorphic factors for the bundles E and E_1 respectively,
$$(*) \quad h(z+\alpha)J(\alpha, z) = J_1(\alpha, z)h(z), \quad \alpha \in L, \quad z \in V.$$
Conersely, given a holomorphic mapping $h : V \to M_{s,r}$ satisfying the above condition $(*)$, then h determines a bundle homomorphism $\pi : V \times C^r \to V \times C^s$ which comutes with the action of L and hence determines a bundle homomorphism $\pi : E \to E_1$.

In summary, it has been shown that there exists a one-to-one correspondence between the set of all isomorphic classes of vector bundles of rank r over a complex torus $T = V/L$ and the set of all equivalence classes of automorphic factors of rank r for the lattice L. Therefore the problem of classifying holomorphic vector bundles of rank r over a complex torus $T = V/L$ corresponds to that of classifying automorphic factors of rank r for the lattice L. In general, the determination of all automorphic factors for the lattice L is a difficult problem. However the classification of atomorphic factors of rank one was completely done by Appel. This problem is equivalent to the computation of the group $H^1(T, \mathcal{O}^*) \cong \mathrm{Pic}\, T$ of isomorphic classes of holomorphic line bundles over T. Now we give an explanation in detail.

Let $T = V/L$ be a g-dimensional complex torus and let H be a Riemann form for the complex torus T. A map $\chi : L \to C_1^* = \{z \in C \mid |z| = 1\}$ is said to be a *semi-character* of L with respect to $E = \mathrm{Im}\, H$ if it satisfies the codition
$$\chi(\alpha + \beta) = \chi(\alpha)\chi(\beta)\exp\{i\pi E(\alpha, B)\}, \quad \alpha. \beta \in I.$$
We define the mapping $J_{H,\chi} : L \times V \to C^*$
$$J_{H,\chi}(\alpha, z) = \chi(\alpha)\exp\left\{\pi H(z, \alpha) + \frac{\pi}{2}H(\alpha, \alpha)\right\},$$
where $\alpha \in L$, $z \in V$ and χ denotes a semi-character of L with respect to $E = \mathrm{Im}\, H$. Then it is easily shown that $J_{H,\chi}$ is an autom-

orphic factor of rank one for the lattice L. We denote by $L(H, \mathfrak{X})$ the holomorphic line bundle over $T = V/L$ defined by the above automorphic factor $J_{H,x}$ for L. We can easily show that

$$L(H_1, \mathfrak{X}_1) \otimes L(H_2, \mathfrak{X}_2) \cong L(H_1 + L_2, \mathfrak{X}_1 \mathfrak{X}_2).$$

Therefore the set of all $L(H, \mathfrak{X})$ forms a group under tensor product. Now we have the following theorem.

THEOREM (Appell-Humbert). *Any holomorphic line bundle over a complex torus $T = V/L$ is isomorphic to an $L(H, \mathfrak{X})$ for a uniquely determined Riemann form H for T and a uniquely determined semi-character $\mathfrak{X}$.*

The Chern class of $L(H, \mathfrak{X})$ is given $E = \operatorname{Im} H \in H^2(T, Z)$. Since $H(x, y) = E(ix, y) + iE(x, y)$ for all $x, y \in V$, according to the above theorem, a holomorphic line bundle over T must be of the form $L(H, \mathfrak{X})$, where $\mathfrak{X}$ is a character of L. Thus group $\operatorname{Pic}^0(T)$ of holomorphic line bundles with Chern class zero is isomorphic to the group $\operatorname{Hom}(L, C_1^*)$ of all characters of L. In fact, $\operatorname{Pic}^0(T) \cong \operatorname{Hom}(L, C_1^*)$ has the structure of a complex torus. By the following exact sequence of sheaves

$$0 \longrightarrow Z \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}^* \longrightarrow 0,$$

we have an ismorphism $\operatorname{Pic}^0(T) \cong H^1(T, \mathcal{O})/\operatorname{Im} H^1(T, Z)$. Now we have $H^1(T, \mathcal{O}) = H_{0,1}(T) = \overline{V}^*$. We know that the image of $H^1(T, Z)$ in $H^1(T, \mathcal{O})$ is the set of all conjugate linear functionals on V whose real part is half-integral on L. Thus $\operatorname{Pic}^0(T)$ is a complex torus, called the dual complex torus of T and is denoted by $\hat{T}$. If T is an abelian variety, so is $\hat{T}$ and it is called the dual abelian variety of T.

We now compute the curvature of a holomorphic line bundle $F = L(H, \mathfrak{X})$ over $T = V/L$. We fix a hermitian structure h on F. We pull back h to $\pi^* F = \tilde{F}$ to obtain an hermitian structure on $\tilde{F} = V \times C$. We may consider $\tilde{h}$ as a positive function on V invariant under L. That is, $\tilde{h}$ satisfies

$$\tilde{h}(z) = |J_{H,x}(\alpha, z)|^2 \tilde{h}(z + \alpha), \quad \alpha \in L, \quad z \in C.$$

Then the connection form $\tilde{\omega} = \partial \log \tilde{h}$ and the curvature form $\tilde{\Omega} = \bar{\partial}\partial \log \tilde{h}$ are given by

$$\tilde{\omega}(z) = \tilde{\omega}(z + \alpha) + \partial \log J_{H,x}(\alpha, z),$$

$$\tilde{Q}(z) = \tilde{Q}(z+\alpha).$$

Thus the curvature form of F is an ordinary 2-form on T. Mutiplying h by a suitable C^∞. positive function on T. we may assume that the curvature form of $F = L(H, \chi)$ is a harmonic $(1, 1)$-form on T. Since $H^2(T, C) = \Lambda^2 V^* \oplus V^* \otimes \bar{V}^* \oplus \Lambda^2 \bar{V}^*$, a harmonic form on T has constant coefficients with respect to the natural coordinates $z_1, \cdots, z_g$ in V. In particular, the curvature Ω of $F = L(H, \chi)$ is given by

$$\Omega = \sum c_{kl} dz_k \wedge d\bar{z}_l,$$

where c_{ki} are constant.

4. Line Bundles over Complex Tori

In this section, we will review line bundles over complex tori. For details, we refer to Mumford [1] and Yang [1].

LEMMA 4.1. *Let E be a holomorphic vector bundle over a complex torus T. Let $T_x (x \in T)$ be the translation of T by x. Then $T_x^*(E)$ and E have the same Chern classes.*

LEMMA 4.2. *Let E be a holomorphic vector bundle f rank r over a complex torus $T = V/L$. Let J be the automorphic factor for the bundle E. Let $x \in T$. Then an automorphic factor J_x for the bundle $T_x^*(E)$ is given by*
$$J_x(\alpha, z) = J(\alpha, z+a), \quad \alpha \in L, \quad z \in V,$$
where a is an element of V such that $\pi(a) = x$.

DEFINITION 4.3. A holomorphic vector bundle E over a complex torus T is said to be *homogeneous if for all $x \in T$, $T_x^*(E) \cong E$.*

LEMMA 4.4. *For a holomorphic vector bundle E of rank r over a complex torus $T = V/L$, the following conditions are equivalent :*
  (1) *E is homogeneous,*
  (2) *E is defined by a representation $\rho : L \to GL(r ; C)$,*
  (3) *E admits a flat connection,*
  (4) *E is a flat vector bundle.*

LEMMA 4.5. *If E is a homogeneous vector bundle of rank r over complex torus T, then $c_k(E) = 0$ for $k \geq 1$, where $c_k(E)$ denotes the*

k-th Chern class of E.

REMARK 4.6. If *rank* $E=r\geq2$, the converse of Lemma 3.5 does not hold. A counter example was given by Oda [2]. However if $r=1$, the converse of Lemma 4.5 is true. For the proof, we refer to Yang [1].

DEFINITION 4.7. Let F be a holomorphic line bundle over a complex torus $T=V/L$. We set $K(F)=\{x\in T|\ T_x^*(F)\cong F\}$. And we define the map $\phi_F : T\to\hat{T}=\mathrm{Pic}^0(T)$ by.
$$\phi_F(x) = T_x^*(F)\otimes F^{-1}, \quad x\in T.$$

REMARKS 4.8. (1) IF F is ample, then $K(F)$ is a finite subgroup of T.

(2) ϕ_F is an isogeny and the set $K(F)$ is nothing but the kernel of ϕ_F.

DEFINITION 4.9. Let n be an integer, We define $n_T : T\to T$ by
$$n_T(x) = nx, \quad x\in T.$$
The map n_T is called the *multiplication by n*. The following lemma shows that n_T is an isogeny of degree n^{2g} if T is an abelian variety of dimension g.

LEMMA 4.10. *Let F be an ample line bundle over an abelian variety A. Then*
$$n_T^*(F) = F^{n^2}\otimes F_0 \quad for\ some\ F_0\in\hat{A}.$$
And the degree of n_T is n^{2g}.

For the proof of the above lemma, we refer to Mumford [1], p. 59, p. 63 or Yang [1], p. 14.

THEOREM OF SQUARE. *Let F be an ample line bundle over an abelian variety $A=V/L$. Then for any $x, y\in A$,*
$$T_{x+y}^*(F)\otimes F\cong T_x^*(F)\otimes T_y^*(F).$$

COROLLARY 1. $\phi_F : A\to\hat{A}=\mathrm{Pic}^0(A)$ *is a homorphism of A to $\hat{A}$.*

COROLLARY 2. *Let $F_1\ F_2$ be two ample line bundles over an abelian variety A. Then*
$$\phi_{F_1\otimes F_2}=\phi_{F_1}+\phi_{F_2}.$$

COROLLARY 3. $\phi_{T_x^*(F)} = \phi_F$ for all $x \in A$.

DEFINITION 4.11. An ample line bundle F over an abelian variety A is said to be *symmetric* if $(-1)_A^* F \cong F$.

LEMMA 4.12. *If F is an ample, symmetric line bundle over an abelian variety A, then $F \otimes (-1)_F^* F$ is also an ample, symmetric line bundle over A.*

PROPOSITION 4.13. *Let A be an abelian variety of dimension g. Then we have*

(1) *$F \in \hat{A}$ if and only if $\phi_F = 0$.*

(2) *Let $f, g : A \to A$ be holomorphic maps. If $F \in \hat{A}$, then $(f+g)^*(F) \cong f^*(F) \otimes g^*(F)$.*

(3) *If $F \in \hat{A}$, then $n_A^*(F) \cong F^n$.*

(4) *If $F \in H^1(A, \mathcal{O})$ has finite order, then $\phi_F \in \hat{A}$.*

(5) *If F is an ample line bundle over A, then ϕ_F is surjective.*

(6) *If $F \in \hat{A}$ and F is not trivial, then $H^i(A, F) = 0$ for all i.*

In the previous section, we introduced the line bundle $L(H, \mathcal{X})$ over complex torus T. Using Lemma 3.2, we have

PROPOSITION 4.14. Let $a \in V$ and $x = \pi(a)$. Then
$$T_x^* L(H, \mathcal{X}) \cong L(H, \mathcal{X} D_a),$$
where $D_a(\alpha) = exp\{2\pi i A(a, \alpha)\}$.

REMARKS 4.15. If $a \in V$, then $\phi_{L(H,\mathcal{X})}(\pi(a)) \cong L(0, D_a)$. Thus we have
$$K(L(H, \mathcal{X})) = L^\perp / L,$$
where $L^\perp = \{v \in V \mid A((\alpha, v) \in Z \text{ for all } \alpha \text{ on } L\}$. Therefore we have

(1) $\qquad L(H, \mathcal{X}) \in \mathrm{Pic}^0(T) = \hat{T}$
$$\Longleftrightarrow K(L(H, \mathcal{X})) = T \Longleftrightarrow L^\perp = V$$
$$\Longleftrightarrow A = 0 \Longleftrightarrow H = 0.$$

(2) $\qquad K(L(H, \mathcal{X}))$ is a finite subgroup of T
$$\Longleftrightarrow L^\perp / L \text{ is finite} \Longleftrightarrow L^\perp \text{ is a lattice}$$
$$\Longleftrightarrow A \text{ is nondegenerate} \Longleftrightarrow H \text{ is nondegenerate}.$$

Let $T = V/L$ be an abelian variety of dimension g. From the exact sequence
$$0 \longrightarrow Z \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}^* \longrightarrow 0,$$

$T \times \hat{T}$.

Now we will describe the Poincaré bundle over $T \otimes \hat{T}$ explicitly. We define an Hermitian form H on $V \otimes \bar{V}^*$ by

$$H((z_1, l_1), (z_2, l_2)) = \overline{l_2(z_1)} + l_1(z_2),$$

where $z_1, z_2 \in V$, and $l_1, l_2 \in V^*$. We also define the map $\chi : L \times \hat{L} \to C_1^*$ by

$$\chi(\alpha, l) = \exp\{-\pi i \operatorname{Im} l(\alpha)\}, \quad \alpha \in L, \quad l \in \hat{L}.$$

Then χ is a semicharacter of $L \times \hat{L}$ with respect to H. That is,

$$\chi((\alpha + \beta, l + \hat{l})) = \chi(\alpha, l)\chi(\beta, \hat{l})\exp\{i\pi E((\alpha, l), (\beta, \hat{l}))\},$$

where $\alpha, \beta \in L$, $l, \hat{l} \in \hat{L}$, and $E - \operatorname{Im} H$. Then the line bundle $L(H, \chi)$ over $T \times \hat{T}$ defined by the Hermitian form H and the semi-character χ of L is the Poincaré line bundle over $T \times \hat{T}$. In fact, the corresponding automorphic factor $J : L \times \hat{T} \to C^*$ for $L(H, \chi)$ is given by

$$J((\alpha, \hat{l}), (z, l)) = \chi(\alpha, \hat{l})\exp\{\pi H((z, l), (\alpha, \hat{l})) + \frac{\pi}{2}H((\alpha, \hat{l}), (\alpha, \hat{l}))\},$$

where $\alpha \in L$, $\hat{l} \in \hat{L}$, $z \in V$ and $l \in \bar{V}^*$. The line bundle $L(H, \chi)|_{T \times \pi(l)}$ over T corresponding to a point $\pi(l) \in \hat{T}(l \in \bar{V}^*)$ is defined by a flat automorphic factor $J_l : L \times V \to C^*$ given by

$$J_l(\alpha, z) = \exp\{\pi l(\alpha)\}, \quad \alpha \in L, \quad z \in V.$$

However

$$h(z + \alpha)J_l(\alpha, z) = \exp\{2\pi i \operatorname{Im} l(\alpha)\}h(z),$$

where $\alpha \in L$, $z \in V$ and $h(z) = \exp\{-\pi \overline{l(z)}\}$ is holomorphic in z. Thus we obtain the isomorphism

$$L(H, \chi)|_{T \times \pi(l)} \cong L(0, \chi_l),$$

where $\chi_l(\alpha) = \exp\{2\pi i l(\alpha)\}$ is a semicharacter of $L(\alpha \in L)$. We note that if $l = 0$, $L(H, \chi)|_{T \times \hat{e}} \cong L(0, 1) \cong \mathcal{O}_T$. Clearly

$$L(H, \chi)|_{e \times \hat{T}} \cong L(0, 1) \cong \mathcal{O}_{\hat{T}}.$$

By the uniqueness, $L(H, \chi)$ must be the Poincaré line bundle over $T \times \hat{T}$.

5. Projectively flat vector bundles

In his paper ([1]), Mukai characterized semihomogeneous vector bundles over an abelian variety and showed that a simple semihomogeneous vector bundle is Gieseker-stable. He dealt with those vector bundles algebraically. In the analytical point of view, those

vector bundles corresponds to the projectively flat vector bundles. In this section, we characterize those vector bundles analytically and study the properties of them.

First we give some definitions.

DEFINITION 5.1. Let $\mathscr{E}$ be a torsion-free coherent sheaf over a compact Kaehler manifold (X, g) of dimension n. Let ω be its Kaehler form. It is a real positive closed $(1, 1)$-form on X. Let $c_1(\mathscr{E})$ be the first Chern class of $\mathscr{E}$. It is represented by a real closed $(1, 1)$-form on X. The degree of $\mathscr{E}$ is defined to be

$$\deg(\mathscr{E}) = \int_X c_1(\mathscr{E}) \wedge \omega^{n-1}.$$

The degree/rank ratio or slope $\mu(\mathscr{E})$ is defined to be

$$\mu(\mathscr{E}) = \deg(\mathscr{E}) / \mathrm{rank}(\mathscr{E}).$$

A coherent sheaf $\mathscr{E}$ over a compact Kaehler manifold (X, g) is said to be *stable*(resp. *semistable*) if for every coherent(nontrivial) proper subsheaf $\mathscr{F}$ of lower rank, $\mu(\mathscr{F}) < \mu(\mathscr{E})$ (resp. $\leq$).

DEFINITION 5.2. Let L be an ample line bundle over a compact Kaehler manifold X. For a coherent sheaf $\mathscr{E}$ over X, we denote by $\mathscr{E}(k)$ the sheaf $\mathscr{E} \otimes L^k$ and by $\chi(\mathscr{E}(k))$ the Hilbert polynomial. We define $P_{\mathscr{E}}$ by

$$P_{\mathscr{E}}(k) = \frac{\chi(\mathscr{E}(k))}{\mathrm{rank}(\mathscr{E})}$$

$\mathscr{E}$ is said to be *Gieseker-stable*(resp. *Gieseker-semistable*) if for each proper subsheaf $\mathscr{F}$ of $\mathscr{E}$, we have

$$P_{\mathscr{F}}(k) < P_{\mathscr{E}}(k) \quad (\text{resp. } P_{\mathscr{F}}(k) \leq P_{\mathscr{E}}(k))$$

for all $k \gg 0$.

REMARK 5.3. (1) A holomorphic vector bundle E over X is said to be stable (resp. semistable) if the sheaf $\mathscr{O}(E)$ of germs of holomorphic sections is stable (resp. semistable). Similarly we can say for the Gieseker stability.

(2) It is known that if a coherent sheaf $\mathscr{E}$ is stable (resp. semistable), it is Gieseker-stable (resp. Gieseker-semistable).

The concept of stability is the algebraic geometrical concept. Kobayashi [1] interpreted the concept of stability differential

geometrically.

DEFINITION 5.4. Let g be a Kaehler metric on a compact Kaehler manifold X. Then we define $tr_g : A^{1,1}(\mathrm{End}(E)) \to A^0(\mathrm{End}(E))$ as follows: For a section $F = (F_\alpha^\beta) \in A^{1,1}(\mathrm{End}(E))$,

$$tr_g F = (\sum g^{j\bar{k}} F_{\alpha j\bar{k}}^\beta)_{1 \leq \alpha, \beta \leq n} = \sum_{j,k} g^{j\bar{k}} F_{j\bar{k}},$$

where $F_\alpha^\beta = F_{\alpha j\bar{k}}^\beta dz_j \wedge d\bar{z}_k$ and $F_{j\bar{k}} = (F_{\alpha j\bar{k}}^\beta)_{1 \leq \alpha, \beta \leq r}$. A holomorphic vector bundle of rank r over a compact Kaehler manifold (X, g) is said to be *Hermitian-Einstein* if there exists an hermitian metric h for which the Hermitian curvature satisfies

$$tr_g F = \mu I,$$

where I is the identity endomorphism of E and μ is a constant.

Kobayashi [1] obtained the following differential geometrical criterion for stability.

THEOREM (Kobayashi). *An indecomposable Hermitian-Einstein vector bundle over a compact Kaehler manifold is stable.*

The fact that the converse of the above theorem is also true was proved by Uhlenbeck and Yau [1].

DEFINITION 5.5. Let E be a holomorphic vector bundle of rank r over a compact Kaehler manifold X and P its associated principal $GL(r ; C)$-bundle. Then $\hat{P} = P/C^* I_r$ is a principal $PGL(r ; C)$-bundle. We say that E is *projectively flat* when $\hat{P}$ is provided with a flat sturcture.

Mukai [1] introduced the notion of semihomogeneous vector bundles over a complex torus.

DEFINITION 5.6. A holomorphic vector bundle E over a complex torus T is said to be *semihomogeneous* if for each $x \in T$, there exists a line bundle F over T such that

$$T_x^*(E) \cong E \otimes F,$$

where T_x is the translation of T by x.

LEMMA 5.7. *Let E be a semihomogeneous vector bundle of rank r over an abelian variety A. Then the vector bundle $(r_T)^*(E) \otimes$*

$(det\ E)^{-r}$ *is homogeneous, where det E denotes the determinant line bundle of E.*

Proof. $L=(r_T)^*(E)\otimes(det\ E)^{-r}$. Then we have
$$det\ L=(r_T)^*(det\ E)\otimes(det\ E)^{-r^2}.$$
By Lemma 4.10 in the preceding section, $c_1(det\ L)=0$ and hence det L is homogeneous. It is clear that L is semihomogeneous. Since $T_x^*(L)\cong L\otimes F$ for some $F\in Pic^0(T)$, we obtain
$$T_{rx}^*(L)\cong H\otimes F^r,$$
$$T_x^*(det\ L)\cong det\ L\otimes F^r.$$
Therefore
$$T_{rx}^*(L)\cong L\otimes T_x^*(det\ L)\otimes(det\ L)^{(-1)}$$
$$\cong L(\text{because det }L\text{ is homogeneous}).$$
Since A is divisible, L is homogeneous.

PROPOSITION 5.8. *Let E be a semihomogeneous vector bundle of rank r over an abelian variety A of dimension g. Then*
$$\chi(E)=r^{1-g}\chi(det\ E),$$
where $\chi(E)$ denotes the Euler-Poincaré characteristic of E.

Proof. Since $(r_T)^*(E)\cong(det\ E)^r\otimes F$ for some homogeneous vector bundle F, we have
$$r^{2g}\chi(E)=\chi((r_T)^*(E))$$
$$=\chi((det\ E)^r\otimes F)\quad(\text{because }c_k(F)=0\text{ for }k\geq1)$$
$$=r\chi((det\ E)^r)$$
$$=r^{g+1}\chi(det\ E).$$
Hence $\chi(E)=r^{1-g}\chi(det\ E)$.

Mukai [1] and Oda [2] showed the following theorem.

THEOREM 5.9 (Mukai, Oda). *Let E be a simple vector bundle over an abelian variety A of dimension g. Then the following are equivalent:*

(1) $dim_C H^1(A, End(E))=g,$

(2) $dim_C H^j(A, End(E))=\binom{g}{j},$

(3) *E is semihomogeneous,*

(4) *There exist an isogeny $f:B\to A$ and a line bundle L on an abelian variety B such that $E=f_*(L)$.*

It is easily seen from the definition that semihomogenous vector bundles over a complex torus are projectively flat. Hano [1] showed that an automorphic factor for a projectively flat vector bundle E over a complex torus $T=V/L$ is given by the following form

$$(*) \quad J(\alpha, z) = G(\alpha)\exp\left\{\frac{\pi}{r}H(z, \alpha) + \frac{\pi}{2r}H(\alpha, \alpha)\right\}, \quad \alpha \in L, \quad z \in V,$$

where (i) H is a Riemann form for $T=V/L$,
  (ii) $G : L \rightarrow GL(r ; C)$ is a semirepresentation of L in the sense:

$$G(\alpha+\beta) = G(\alpha)G(\beta)\exp\left\{\frac{i\pi}{r}E(\beta, \alpha)\right\}, \quad E = \operatorname{Im} H.$$

Using Lemma 4.2 in the previous section, we can show that a projectively flat vector bundle over a complex torus is semihomogeneous. Thus the notion of semihomogeneous vector bundles is the same of projectively flat vector bundles.

LEMMA 5.10. *Let* $f : \tilde{T} = \tilde{V}/\tilde{L} \rightarrow T = V/L$ *be an isogeny and let* E *(resp. F) be a projectively flat vector bundle of rank* r *over* T *(resp. $\tilde{T}$). Then* $f^*(E)$ *(resp. $f_*(F)$) is also projectively flat.*

Proof. f lifts to the linear map $f : V \rightarrow V$. Then the automorphic factor J^* for $f^*(E)$ is given by

$$\tilde{J}(\tilde{\alpha}, \tilde{z}) = G(f(\tilde{\alpha})) \exp\left\{\frac{\pi}{r}H(f(\tilde{z})), f(\tilde{\alpha})) + \frac{\pi}{2r}H(f(\tilde{\alpha}, f(\tilde{\alpha}))\right\},$$

where $\tilde{\alpha} \in \tilde{L}$, $z \in \tilde{V}$. thus $f^*(E)$ is semihomogeneous. We leave to the reader the case of $f_*(E)$.

PROPOSITION 5.11. *Let E be a hermitian vector bundle of rank* r *over an abelian surface A with $c_2(End(E))=0$. Then the following are equivalent:*
  (1) *E is simple,*
  (2) *E is simple and semihomogeneous,*
  (3) *E is simple and projectively flat,*
  (4) *E admits an indecomposable Hermitian–Einstein vector bundle over A,*
  (5) *E is stable,*
  (6) *E is Gieseker–stable.*

Proof. $(1) \Rightarrow (2)$:

$$\chi(\mathrm{End}(E)) = 2h^0(A, \mathrm{End}(E)) - h^1(A, \mathrm{End}(E))$$
$$= 2 - h^1(A, \mathrm{End}(E)) \,(\text{because } E \text{ is simple}).$$

But by the Riemann-Roch theorem,
$$\chi(\mathrm{End}(E)) = -c_2(\mathrm{End}(E)) = 0.$$

Thus we have $h^1(A, \mathrm{End}(E)) = 2$. By Theorem 5.10, E is semihomogeneous. The remaining ones follows from Kobayashi's theorem and the fact that a Gieseker-stable vector bundle is simple.

Since a simple projective flat hermitian vector bundle E over a complex torus T admits an Hermitian-Einstein structure, E is Gieseker-stable by Kobayashi's theorem (or Mukai [1]). Thus E has a filtration
$$0 = E_0 \subset E_1 \subset \cdots \subset E_k = E$$
such that $F_i = E_i/E_{i-1}$ is Gieseker-stable, and $P_{F_i} = P_E$ for all $i = 1, 2, \cdots, k$ (Gieseker [1]).

An automorphic factor J for E is given by the form (∗). Now we calculate the curvature form Ω of E. We let $\pi : V \to T = V/L$. We choose an open covering $\{U_i\}$ of T with the following property: U_i are connected and each connected component of $\pi^{-1}(U_i)$ are mapped homeomorphically onto U_i by π. For U_i we choose a connected component $\tilde{U}_i$ of $\pi^{-1}(U_i)$. Then we have $\pi^{-1}(U_i) = \bigcup_{\alpha \in L} T_\alpha \tilde{U}_i$, where $T_\alpha : V \to V$ is the translation of V by $\alpha \in L$. We let
$$\rho_i : U_i \longrightarrow \tilde{U}_i$$
be the inverse of the homeomorphism $\pi : \tilde{U}_i \to U_i$. For each pair (i, j) of indices such that $U_i \cap U_j \neq \phi$, there exists a unique $\sigma_{ji} \in L$ such that
$$\rho_i(x) = \rho_j(x) + \sigma_{ji}$$
for all $x \in U_i \cap U_j$. For all $x \in U_i \cap U_j$, we let
$$g_{ij}(x) = J(\sigma_{ji}, \rho_j(x)).$$

Then $g_{ij} : U_i \cap U_j \to GL(r, C)$ is a holomorphic map and $\{g_{ij}\}$ is a system of transition functions of the vector bundle E over a complex torus T.

We take a basis of V and identify V with C^g and write
$$H(v, w) = \sum_{a, b=1}^{g} h_{a\bar{b}} v_a \bar{w}_b.$$

Then we have

$$g_{ij} = G(\sigma_{ji}) \exp\left\{ \frac{\pi}{r} H(\rho_j, \sigma_{ji}) + \frac{\pi}{2r} H(\sigma_{ji}, \sigma_{ji}) \right\}.$$

We let

$$z_a^{(i)} = v_a \circ \rho_i$$

for each i. Then $\{z_1^{(i)}, \cdots, z_g^{(i)}\}$ are local coordinates of T on U_i and we obtain

$$dz_a^{(i)} = dz_a^{(j)} \quad \text{on } U_i \cap U_j$$

Let ζ_a be the holomorphic 1-form on T such that $\pi^* \zeta_a = dv_a$ Then we have

$$\zeta_a = dz_a^{(i)}$$

on each U_i. We get

$$g_{ij}^{-1} dg_{ij} = \left\{ \frac{\pi}{r} \sum_{a,b} h_{a\bar{b}} (\bar{\sigma}_{ji})_b \zeta_a \right\} \cdot I_r,$$

where I_r is the $r \times r$ identity matrix. We let

$$\omega_i = - \left\{ \frac{\pi}{r} \sum_{a,b} h_{a\bar{b}} \bar{z}_b^{(i)} \right\} \cdot I_r$$

on each U_i. Then it is easy to show that $\omega = \{\omega_i\}$ is a connection form. The curvature form $\Omega = \{\Omega_i\}$ is the system of 2-forms such that

$$\Omega_i = d\omega_i + \omega_i \wedge \omega_i$$

on U_i. But we have $\omega_i \wedge \omega_i = 0$ and hence $\Omega_i = d\omega_i$. Then we have

$$\Omega_i = \left\{ \frac{\pi}{r} \sum_{a,b} h_{a\bar{b}} \zeta_a \wedge \bar{\zeta}_b \right\} \cdot I_r$$

on U_i and since the left hand side is globally defined, we have globally

$$\Omega = \left\{ \frac{\pi}{r} \sum_{a,b} h_{a\bar{b}} \zeta_a \wedge \bar{\zeta}_b \right\} \cdot I_r.$$

The total Chern class $c(E)$ is given by

$$c(E) = \det\left(I_r - \frac{1}{2\pi i} \Omega \right)$$

$$= \left(1 + \frac{i}{2r} \sum_{a,b} h_{a\bar{b}} \zeta_a \wedge \bar{\zeta}_b \right)^r.$$

Let E be a holomorphic vector bundle of rank r over a complex torus T. We assume that the total Chern class $c(E)$ of E is

$$c(E) = \left(1 + \frac{c_1(E)}{r} \right)^r.$$

We know that $c(E)$ is given by

$$c(E) = \det\left(I_r - \frac{\Omega}{2\pi i}\right),$$

where Ω is the curvature form of E. If we write $\Omega = (\Omega_j^i)$, the k-th Chern class $c_k(E)$ of E is given by

$$c_k(E) = \binom{r}{k}\frac{1}{r^k}c_1(E)^k$$

$$= \frac{(-1)^k}{(2\pi i)^k k!}\sum \delta_{i_1 \cdots i_k}^{j_1 \cdots j_k}\Omega_{j_1}^{i_1}\wedge\cdots\wedge\Omega_{j_k}^{i_k}.$$

By a tedious calculation, we know that Ω is of the form

$$\Omega = \delta I_r,$$

where δ is a 2-form on T. Hence E is projectively flat.

In summary, we have

THEOREM 5.12. *Let E be a holomorphic vector bundle over rank r over a complex torus $T = V/L$. Then the following conditions are equivalent:*

(1) *E is semihomogeneous,*

(2) *E is projectively flat,*

(3) *The total Chern class $c(E)$ of E is given by*

$$c(E) = \left(1 + \frac{c_1(E)}{r}\right)^r,$$

(4) *The Automorphic factor J for E is given by*

$$J(\alpha, z) = G(\alpha)\exp\left\{\frac{\pi}{r}H(z, \alpha) + \frac{\pi}{2r}H(\alpha, \alpha)\right\}, \quad \alpha \in L, \ z \in V,$$

where $G : \to GL(r ; C)$ is a semirepresentation of L and H is a Riemann form for T. Furthermore, if E is simple, (1), (2), (3) and (4) are equivalent to

(5) *$dim_c H^j(T, End(E)) = \binom{g}{j}$ for all $j = 1, 2, \cdots, n$.*

(6) *$H^j(T, \mathscr{O}) \cong H^j(T, End(E))$ for all j,*

(7) *There exists an isogeny $f : \tilde{T} \to T$ and a line bundle L on $\tilde{T}$ such that $E = f_*(L)$.*

6. Heisenberg Group $\mathscr{G}(E)$

Throughout this section X is assumed to be an abelian variety of dimension g over the field of complex numbers. We recall that

n_x is the multiplication by n for an integer n (see Definition 4.9).

(6.1) Let E be a holomorphic vector bundle over X. We define $H(E)$ and $\mathscr{G}(E)$ as follows:

$H(E) = \{x \in X \mid E \cong T_x^*(E)\}$,

$\mathscr{G}(E) = \{(x, \varphi) \mid x \in H(E)$ and φ is an isomorphism of E onto $T_x^*(E)\}$. $\mathscr{G}(E)$ is a group. In fact, let (x, φ), (y, ψ) be elements of $\mathscr{G}(E)$. Then the composition $T_x^*(\psi \circ \varphi)$:

$$E \longrightarrow T_x^*(E) \longrightarrow T_x^*(T_y^*(E)) = T_{x+y}^*(E)$$

is an isomorphism of E and T_{x+y}^*. We define the multiplication

$$(y, \psi) \circ (x, \varphi) = (x+y, T_x^*(\psi \circ \varphi)).$$

It is easy to check that the set $\mathscr{G}(E)$ forms a group under the above multiplication. And we have the following exact sequence

$$1 \longrightarrow \mathrm{Aut}(E) \longrightarrow \mathscr{G}(E) \longrightarrow H(E) \longrightarrow 0.$$

(6.2) Let L be an ample line bundle over an abelian variety X. We recall the basic results about an ample line bundle (Mumford [1]).

(I) $H(L)$ is finite and $H^0(X, L^n) \neq 0$ for all $n > 0$.

(II) If dim $X = g$, then there exists a positive integer d such that

$$\dim_C H^0(X, L^n) = dn^g \text{ for all } n \geq 1,$$
$$\dim_C H^i(X, L^n) = 0 \text{ for all } n \geq 1, \ i \geq 1.$$

The integer d is called the *degree* of L

(III) Let $\hat{X}$ be the dual abelian variety of X. Let $\Lambda(L) : \to \hat{X}$ be the homomorphism define by $\Lambda(L)(x) = T_x^*(L) \otimes L^{-1}$ for all $x \in X$. Then we have

$$d^2 = |\chi(L)|^2 = \text{the degree of } \Lambda(L) = |H(L)|.$$

(IV) For all integers n,

$$(n_x)^*(L) \cong L^{\frac{n^2+n}{2}} \otimes (-1_x)^*(L)^{\frac{n^2-n}{2}}.$$

(6.3) Let E be a stable ample vector bundle over X. Then E is simple and hence we have the following extension:

$$1 \longrightarrow C^* \longrightarrow \mathscr{G}(E) \longrightarrow H(E) \longrightarrow 0.$$

We note that $H(E)$ is a finite abelian group and that C^* is contained in the center of $\mathscr{G}(E)$. The extension defines the following invariant : Given $x, y \in H(E)$, we let $x, y \in \mathscr{G}(E)$ lie over x, y. We set

$$e^E(x, y) = \hat{x}\hat{y}\hat{x}^{-1}\hat{y}^{-1}.$$

It is obvious that this is well-defined, that $e^E(x, y)$ is an element H of C^*, and that e^E is a skew-symmetric bilinear pairing from $H(E)$ to C^*. A subgroup K of $\mathscr{G}(E)$ is said to be a *level subgroup* if $\hat{K} \cap C^* = (0)$, i.e., $\hat{K}$ is isomorphic to its image in $H(E)$. For all subgroup K of $H(E)$, there exists a level subgroup $\hat{K}$ over K if and only if e^E is trivial on K. If e^E is degenerate, then there exists a subgroup K such that e^E is trivial on $\hat{K}$ and such that $|K|^2 > |H|$. Hence there exists a level subgroup $\hat{K}$ of order $> |H|^{\frac{1}{2}}$. We now define $U_z : H^0(X, E) \to H^0(X, E)$ by
$$U_z(s) = T^*_{-x}(\varphi(s)) \quad \text{for all } s \in H^0(X, E),$$
where $z = (x, \varphi) \in \mathscr{G}(E)$. This is an action of the group $\mathscr{G}(E)$ because if $z = (x, \varphi), w = (y, \psi)$, then
$$\begin{aligned}
U_w(U_z(s)) &= T^*_y\{\psi(T^*_x(\varphi s))\} \\
&= T^*_{-x-y}\{T^*_x(\psi(T^*_x(\varphi s)))\} \\
&= T^*_{-x-y}\{T^*_x(\psi)(\varphi s)\} \\
&= U_{(x+y, T^*_x(\psi) \circ \varphi)}(s).
\end{aligned}$$
Also C^* acts on $H^0(X, E)$ by its natural character, that is, $a \in C^*$ acts on $H^0(X, E)$ as multiplication by a. Hence U is the representation of $\mathscr{G}(E)$ on $H^0(X, E)$.

(6.4) Let X be an elliptic curve and let E be an ample indecomposable vector bundle of rank r and of degree d. We assume that r, d are coprime. Then E is stable and hence simple. Thus we have the extension $0 \to C^* \to \mathscr{G}(E) \to H(E) \to 0$ and a level subgroup of $\mathscr{G}(E)$ corresponds to a descent data for E. By Oda [1], there exists an isogeny $f : Y \to X$ of degree r and an ample line bundle L of degree d on Y such that E is isomorphic to the direct image $f_*(L)$ and the intersection of $\ker(f)$ and $\ker(\Lambda(L))$ is 0. Moreover, $d = \dim_c H^0(X, E) = \dim_c H^0(Y, L)$. Since $H(L) \cap \ker(f) = 0$, a non-trivial translation by an element of $H(L)$ induces a nonzero element of $H(E)$. Hence $H(L)$ is a subgroup of $H(E)$. We have $|H(L)| = d^2$, hence $|H(E)| \geq d^2$. There exists a level subgroup of order $\tilde{d} > d$. If we had $|H(E)| > d^2$ then there would exist an isogeny $\varphi : X \to Z$ of degree $\tilde{d}$ and a vector bundle $\tilde{E}$ over Z such that $\varphi^*(\tilde{E}) \cong E$. But we have $d = \chi(X, E) = \tilde{d}\chi(Z, E)$. This is a contradiction. Hence $H(E) = H(L)$ and $\mathscr{G}(E) = \mathscr{G}(L)$. The unique

representation of $\mathscr{G}(E)$ is given by $H^0(X, E)$.

PROPOSIION 6.5. *Let E be a holomorphic vector bundle of rank r over an abelian surface X. Then the following conditions are equivalent:*

(1) *E is a simple homogeneous vector bundle over X,*

(2) *E is an indecomposable projectively flat vector bundle over X,*

(3) *E is stable and satisfies $c_2(End(E)) = 0$,*

(4) *E is Gieseker stable and satisfies $c_2(End(E)) = 0$,*

(5) *E is simple and satisfies $c_2(End(E)) = 0$,*

(6) *$dim_C H^j(X, End(E)) = \binom{g}{j}$ for all $j = 1, 2, \cdots, g$,*

(7) *$H^j(X, Q) \cong H^j(X, End(E))$ for all j,*

(8) *There exists an isogeny $f : Y \rightarrow X$ of abelian surfaces and an ample line bundle M over Y such that $H(M) \cap ker(f) = 0$ and E is isomorphic to the direct image $f_*(M)$,*

(9) *E is simple and, for any ample line bundle L over X and for any sufficiently large integer n, we have the extension of the Heisenberg group*

$$0 \longrightarrow C^* \longrightarrow \mathscr{G}(E \otimes L^n) \longrightarrow H(E \otimes L^n) \longrightarrow 0.$$

such that the pairing $e^{E \otimes L^n}(x, y)$ is nondegenerate and $|H(E \otimes L^n)| = \frac{1}{r^2} h^0(X, (det\, E) \otimes L^{rn})^2$,

(10) *E is simple and the same assertion as in (9) holds for an ample line bundle L and for infinitely many $n > 0$.*

Proof. The equivalence of (1), (2), $\cdots$, (8) follows from Theorem 5.12 and the equation $c_2(End(E)) = -(r-1)c_1(E)^2 + 2rc_2(E)$. In order to prove (8) $\Rightarrow$ (9), we let E be a simple vector bundle of rank r and let L be an ample line bundle over X. By (8), $E \otimes L^n$ is isomorphic to the direct image of $M \otimes f^*(L^n)$. Let $x \in ker(f) \cap H(M \otimes f^*(L^n))$. Then we have

$$M \otimes f^*(L^n) \cong T_x^*(M \otimes f^*(L^n)) \cong T_x^*(M) \otimes T_x^* f^*(L^n)$$
$$\cong T_x^*(M) \otimes f^*(L^n).$$

Hence M is isomorphic to $T_x^*(L)$, i.e., $x \in ker(f) \cap H(M)$. It follows that $x = 0$. As in (6.4), we have the inequality

$$|H(M \otimes f^*(L^n))| \leq |H(E \otimes L^n)|.$$

According to the descent theory, if $M \otimes f^*(L^n)$ is an ample line

bundle, then we get $|H(M \otimes f^*(L^n)) \cong H(E \otimes L^n)$ and $|H(E \otimes L^n)| = h^0(X, E \otimes L^n)^2$. By the Riemann-Roch theorem,

$$h^0(X, E \otimes L^n) = \frac{1}{2} c_1^2(E \otimes L^n) - c_2(E \otimes L^n)$$

$$= \frac{1}{2r} c_1^2(E) + \frac{rn^2}{2} c_1^2(L) + n c_1(E) c_1(L)$$

$$= \frac{1}{2r} c_1^2(E \otimes L^n) = \frac{1}{2r} c_1^2((\det E) \otimes L^{rn})$$

$$= \frac{1}{r} h^0(X, (\det E) \otimes L^{rn}).$$

Thus we have $|H(E \otimes L^n)| = \frac{1}{r^2} h^0(X, (\det E) \otimes L^{rn})$. Obviously (9) implies (10). Thus the proof is complete only if we show that (10) implies (5). We assume that (10) holds. Then there exists an integer n and an ample line bundle L over X such that $H^0(X, E \otimes L^n) \neq 0$, $H^i(X, E \otimes L^n) = 0$, $i = 1, 2$, and $0 \to C^* \to \mathscr{G}(E \otimes L^n) \to H(E \otimes L^n) \to 0$ is a Heisenberg group and $|H(E \otimes L^n)|| = \frac{1}{r^2} h^0(X, (\det E) \otimes L^{rn})^2$. Since $E \otimes L^n$ is simple,

$$\chi(\mathrm{End}(E \otimes L^n)) = h^0(X, \mathrm{End}(E \otimes L^n)) - h^1(X, \mathrm{End}(E \otimes L^n))$$
$$+ h^2(X, \mathrm{End}(E \otimes L^n))$$
$$= 2 - h^1(X, \mathrm{End}(E)) \leq 0.$$

By the Riemann-Roch theorem,

$$\chi(\mathrm{End}(E \otimes L^n)) = -c_2(\mathrm{End}(E \otimes L^n))$$
$$= (r-1) c_1^2(E \otimes L^n) - 2r c_2(E \otimes L^n),$$

Thus $(r-1) c_1^2(E \otimes L^n) - 2r c_2(E \otimes L^n) \leq 0$. Then we have

$$1 \leq h^0(E \otimes L^n) = \frac{1}{2} c_1^2(E \otimes L^n) - c_2(E \otimes L^n)$$

$$\leq \frac{1}{2} c_1^2(E \otimes L^n) - \frac{r-1}{2r} c_1^2(E \otimes L^n)$$

$$= \frac{1}{2r} c_1^2(E \otimes L^n) = |H(E \otimes L^n)|^{\frac{1}{2}}.$$

since $\mathscr{G}(E \otimes L^n)$ is a Heisenberg group and $H^0(X, E \otimes L^n)$ is a representation of $\mathscr{G}(E \otimes L^n)$ in which C^* acts by its natural character, $h^0(E \otimes L^n)$ is divisible by $|H(E \otimes L^n)|^{\frac{1}{2}}$. By the above inequality, we have

$$h^0(X, E \otimes L^n) = |H(E \otimes L^n)|^{\frac{1}{2}}.$$

By the assumtion, we have

$$h^0(X, E \otimes L^n) = \frac{1}{2r} c_1^2(E \otimes L^n).$$

By the way, by the Riemann-Roch theorem, we have

$$h^0(X, E \otimes L^n) = \frac{1}{2} c_1^2(E \otimes L^n) - c_2(E \otimes L^n).$$

Hence we obtain

$$(r-1)c_1^2(E \otimes L^n) - 2rc_2(E \otimes L^n) = 0.$$

So we get the equation

$$c_2(\mathrm{End}(E)) = -(r-1)c_1^2(E) + 2rc_2(E) = 0.$$

REMARK 6.6. Indeed, the classification of projectively flat vector bundles over a complex torus cerresponds to that of representations of the Heisenberg group. Matsushima [2] described the holomorphic vector bundles defined by the representation of the Heisenberg group.

FINAL REMARK 6.7. It is very interesting to characterize the automorphic factors corresponding to stable vector bundles over complex torus. The author believes that a characterization of those automorphic factors will be useful in the study of vector-valued theta functions.

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