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COMPACT KAEHLERIAN MANIFOLDS WITH POSITIVE HYBRID SYMMETRIC CURVATURE OPERATOR

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1. Introduction

Many subjects for the compact Kaehlerian manifold of positive bisectional curvature were investigated from the different points of view ([1], [2], [5], [7], [9] etc.), two of which done by Y.-T. Siu and S.-T. Yau [9] and S. Goldberg and S. Kobayashi [5], assert that the following interesting results:

THEOREM S.-Y. *Every compact Kaehlerian manifold of positive bisectional curvature is biholomorphic to the complex projective space.*

THEOREM G.-K. *A compact Kaehlerian manifold with positive bisectional curvature and constant scalar curvature is isometric to a complex projective space.*

Furthermore, A. Gray [6] proved the following fact:

THEOREM G. *A compact Kaehlerian manifold with nonnegative sectional curvature and constant scalar curvature is locally symmetric.*

Let $R_{A\bar{B}C}{}^{D}$ be the curvature tensor of a Kaehlerian manifold to complex coordinate. E. Calabe and E. Vesentine [3] have dealt with the following two curvature operators of Kaehlerian manifolds:

(A) $\xi_{AB} \to \sum R^{C}{}_{AB}{}^{D}\xi_{CD}$,

(B) $\xi_{A\bar{B}} \to \sum R^{C}{}_{A\bar{B}}{}^{\bar{D}}\xi_{C\bar{D}}$.

K. Ogiue and S. Tachibana [8] showed that a compact Kaehlerian manifold whose curvature operator (A) is positive is biholom-

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orphic to the complex projective space.

The purpose of the present paper is devoted to that compact Kaehlerian manifolds whose curvature operator (B) is positive.

2. Positive hybrid symmetric operator

Let M be an $n=2m$ real dimensional Kaehlerian manifold equipped with a parallel almost complex structure J and a Riemannian metric $\langle\ ,\ \rangle$ which is J-Hermitian. We then have

$$J^2=-I,\quad \langle JX, JY\rangle=\langle X, Y\rangle$$

for any tangent vectors X and Y on M, where I denotes the identity transformation. For $x\in M$ we denote by $T_x(M)$ the tangent space to M at x.

Since J is the parallel tensor field, it is seen that

(2.1) $$\langle R(X, Y)Z, W\rangle=\langle R(X, Y)JZ, JW\rangle,$$
$$S(X, Y)=S(JX, JY)$$

for any X, Y, Z and W in $T_x(M)$, where R and S are denoted respectively by the Riemannian curvature tensor and the Ricci tensor of M.

Let σ be a plane in $T_x(M)$, namely, a real two dimensional subspace of $T_x(M)$. Choosing an orthonormal basis X and Y for σ, we define the holomorphic sectional curvature $K(X, Y)$ of σ by

$$K(X, Y)=\langle R(X, Y)Y, X\rangle.$$

Given two J-invariant planes σ and σ' in $T_x(M)$, the holomorphic bisectional curvature $H(\sigma, \sigma')$ is given by

$$H(\sigma, \sigma')=\langle R(X, JX)JY, Y\rangle,$$

where X is a unit vector in σ and Y a unit vector in σ'. We shall occasionally write $H(\sigma, \sigma)=H(\sigma)$.

A tensor field u of type (0, 2) is said to be hybrid [11], if $u(X, Y)=u(JX, JY)$ for any X and Y in $T_x(M)$.

Let $P_m(c)$ be the m complex dimensional complex projective space with constant holomorphic sectional curvature $c>0$. Then the curvature tensor of $P_m(c)$ satisfies

$$R(u)=\frac{c}{4}\{(t_r u)^2+2t_r u^2\},$$

where $R(u)$ is denoted by

(2.2) $$R(u)=\sum_{i,j,k,l}\langle R(e_i, e_j)e_k, e_l\rangle u(e_i, e_l)u(e_j, e_k)$$

for a hybrid symmetric tensor $u, u(e_i, e_j)$ being the components of u with respect to an orthonormal basis $\{e_i\}$ of $T_x(M)$.

A Kaehlerian manifold will be called of positive hybrid symmetric curvature operator (positive HSC-operator) [10] if there exists a constant $c>0$ satisfying

$$(2.3) \qquad R(u) \geqq \frac{c}{4}\{(t_r u)^2 + 2t_r u^2\}$$

for any hybrid symmetric tensor u of type $(0, 2)$ at each point of M. This condition is equivalent to the positiveness of operator (B) stated in the introduction.

Now, let us put

$$u = X\otimes Y + Y\otimes X + JX\otimes JY + JY\otimes JX$$

for any X and Y in $T_x(M)$. Then it is not hard to see that u is a hybrid symmetric tensor field of type $(2, 0)$.

By a straitforward computation, it is seen that

$$R(u) = 4\{\langle R(X, Y)X, Y\rangle + \langle R(X, JX)JY, Y\rangle + \langle R(X, JY)JX, Y\rangle\},$$

which together with the first Bianchi identity yields

$$(2.4) \qquad R(u) = 8\langle R(X, Y)JY, X\rangle,$$

where we have used (2.1) and (2.2).

Thus, (2.3) is reduced to

$$4\langle R(X, JY)JY), X\rangle \geqq c\{3\langle X, Y\rangle^2 + \langle X, X\rangle\langle JY, JY\rangle - \langle X, JY\rangle^2\},$$

which is equivalent to

$$(2.5) \qquad \langle R(X, JY)JY, X\rangle \geqq \frac{c}{4}\{3\langle X, Y\rangle^2 + \frac{1}{2}\|X\wedge JY\|^2\}$$

Since $c>0$, it follows that

$$\langle R(X, JY)JY, X\rangle > 0$$

for any X and Y in $T_x(M)$ such that $JX \neq Y$. Accordingly, the holomorphic bisectional curvature of M is positive.

Thus, by means of Theorem $S.-Y.$, we have

THEOREM 1. *A compact Kaehlerian manifold with positive hybrid symmetric curvature operator is biholomorphic to the complex projective space.*

Furthermore, using Theorem G.-K., we have

THEOREM 2. *A compact Kaehlerian manifold with positive hybrid* *symmetric curvature operator and constant scalar curvature is isometric to $P_m(c)$.*

We now suppose that c is nonnegative. Then (2.5) implies that the sectional curvature $K(X, Y)$ is nonnegative. Therefore, because of Theorem G., it follows that M is locally symmetric. Thus, we have

THEOREM 3. *Let M be a compact Kaehlerian manifold with nonnegative hybrid symmetric curvature operator. Then M is locally symmetric.*

REMARK. Under the same assumptions as those satated in Theorem 3, M is not always isometric to a $P_m(c)$. For an example, $M=P_{\frac{m}{2}}(c)\times P_{\frac{m}{2}}(c)$ is a Kaehler-Einstein manifold such that M is of nonnegative HSC-operator.

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