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CONTINUITY OF CERTAIN HOMOMORPHISMS OF BANACH ALGEBRAS

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1. Introduction

The continuity problems of homomorphisms from a Banach algebra into a Banach algebra were studied by many mathematicians, yet there remain many open questions on the subject [1], [4]. One of the unsolved problems, perhaps the most important problems in the automatic continuity theory, is whether every homomorphism from a Banach algebra into a noncommutative semisimple Banach algebra is necessarily continuous. In [5] B. E. Johnson showed that any complete norm on a semisimple Banach algebra A is equivalent to the original norm of A. From this result it can be easily seen that a homomorphism from a Banach algebra onto a semisimple Banach algebra is necessarily continuous. Thus the above open problem is reduced to the continuity problems of non surjective homomorphisms. For this kind of homomorphisms the closure of the range is itself a Banach algebra and we consider homorphisms whose range is dense in the codomain.

The continuity of a homomorphism is closely related to the radical of B and many authors investigated continuity conditions in relation with the properties of ideals of Banach algebras. In this note we seek to find conditions which ensure the continuity of a homomorphism. We prove the following theorems.

Let Banach algebras A and B have identities and $\theta : A \to B$ be a homomorphism with dense range such that

(i) for each left ideal J of A there is a left ideal L of B such that $\theta(J) = \theta(A) \cap L$, and

(ii) for each maximal left ideal M of B $\theta(A) \cap M$ is dense in M.

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Then (i) implies the inverse image of a maximal left ideal of B is a maximal left ideal of A, and

(i) and (ii) imply the separating space of θ is contained in the radical of B, thus, in this case θ is continuous if B is semisimple.

2. Preliminaries

Let A and B be Banach algebras over the complex field and let $\theta : A \to B$ be a homomorphism from A into B. By the separating space of $\theta : A \to B$ we mean

$$\mathscr{S}(\theta) = \{b \in B : \text{there is a sequence } \{a_n\} \text{ in } A \text{ such that } a_n \to 0 \text{ in } A \text{ and } \theta(a_n) \to b \text{ in } B\}.$$

It is well-known that $\mathscr{S}(\theta)$ is a two-sided ideal of B if $\theta : A \to B$ has dense range, and $\theta : A \to B$ is continuous if and only if $\mathscr{S}(\theta) = \{0\}$. By the radical of a Banach algebra A we mean the intersection of the kernels of all irreducible representations of A or equivalently the intersection of all maximal modular left ideals of A. Hence, if A has the identity then the radical of A is the interesection of all maximal left ideals.

For a left ideal J of A, $A-J$ will denote the difference space of A modulo J, i. e., $A-J$ is the linear space of all cosets of J. Let a' denote the coset $a+J$ for each a in A, then the operation defined by

$$ab' = (ax)' \text{ for each } a \in A,\ x \in b' \in A-J$$

is clearly a module operation such that $A-J$ becomes a left A-module. With this module multiplication $A-J$ will be called the regular left A-module. On the other hand, let $\mathscr{L}(A-J)$ denote the space of all linear operators. Then the map $\pi : A \to \mathscr{L}(A-J)$ defined by

$$\pi(a)b' = ab' \text{ for each } a \in A,\ b' \in A-J$$

is a representation of A on $A-J$. This representation will be called a left regular represention of A on $A-J$.

3. Continuity of homomorphisms

Let A and B be Banach algebras with the identity elements and let $\theta : A \to B$ be a homomorphism. If θ maps A onto B it can be easiey seen that the inverse image of each maximal ideal of B is a maximal ideal of A [3]. But we can not expect the same result for a homomorphism which maps A into B even if the homomorphism has dense range. However, as we see in the next lemma there is a certain class of homomorphisms with dense range for which inverse image preserves the maximality of ideals. We prove the lemma for left ideals since an obvious modification will prove the same result for right ideals.

LEMMA 1. *Let A and B be Banach algebras with identity and let $\theta : A \rightarrow B$ be a homomorphism with dense range. If for each left ideal J of A there is a left ideal L of B such that $\theta(J) = \theta(A) \cap L$, then the inverse image of a maximal left ideal of B is a maximal left ideal of A.*

Proof. Let M be a maximal left ideal of B. It is obvious that $\theta^{-1}(M)$ is a left ideal of A. Since the range of θ is a dense subalgebra of B θ maps the identity of A to the identity of B, and hence the identity of A does not belong to $\theta^{-1}(M)$. Thus $\theta^{-1}(M)$ is a proper left ideal of A. Suppose that there is a left ideal J of A such that

$$\theta^{-1}(M) \subset J \subset A.$$

It suffices to show that $J = \theta^{-1}(M)$ or $J = A$ to ensure the maximality of $\theta^{-1}(M)$. By hypothesis there is a left ideal L of B such that $\theta(J) = \theta(A) \cap L$. Thus we have

$$\theta(A) \cap L = \theta(J) \supset \theta(\theta^{-1}(M)) = \theta(A) \cap M$$

and hence $L \supset M$. By the maximality of M either $L = M$ or $L = B$. If $L = M$, then

$$\theta^{-1}(M) = \theta^{-1}(L) = \theta^{-1}(\theta(J)) \supset J.$$

Hence, in this case we have $\theta^{-1}(M) = J$. If $L = B$, then $\theta(J) = \theta(A) \cap L = \theta(A)$. If $a \in A$ then there is $j \in J$ such that $\theta(j) = \theta(a)$, thus $a - j$ belongs to the kernel of θ. But the kernel is contained in the ideal J and so $a - j \in J$. Then $a \in J$. Therefore $J = A$ and the proof is completed.

The following elementary fact will be used in the proof of Theorem 3.

LEMMA 2. *Let J be a left ideal of a Banach algebra A with* *identity and let X denote the regular left A-module $A-J$. Then J is a maximal left ideal of A if and only if the left regular representation π of A on X is irreducible.*

Proof. Let $\phi: A \to A-J$ be the quotient map of A onto the difference space $A-J$, and let Y be an A-submodule of X with $Y \neq \{0\}$. Then $L=\{a \in A : \phi(a) \in Y\}$ is a left ideal of A and $J \neq L$ since $Y \neq \{0\}$. If J is maximal then $L=A$. Thus $\phi(e) \in Y$ where e is the identity of A and $Y=X$. Therefore X is irreducible and the left regular representation π is irreducible. Conversely, if J is not maximal it is contained in a maximal left ideal M of X. Then $Z=M-L$ is a left A-submodule of X with $X \neq Z$ and $Z \neq \{0\}$. Therefore X is not irreducible and the left regular representation π is not irreducible.

We need the following well-known theorem [5], [2, Theorem 25.7].

THEOREM (B. E. JOHNSON). *Let θ be an irreducible representation of a Banach algebra A on a normed linear space X such that $\pi(a)$ is continuous on X for each $a \in A$. Then π is continuous on A.*

The following theorem is our main result.

THEOREM 3. *Let A and B be Banach algebras with the identities, and let $\theta: A \to B$ be a homomorphism with dense range. If the following conditions are satisfied, then the separating space $\mathscr{S}(\theta)$ is contained in the radical of B.*

(i) *For each left ideal J of A there is a left ideal L of B such that $\theta(J)=\theta(A) \cap L$.*

(ii) *For each maximal left ideal M of B $\theta(A) \cap M$ is dense in M.*

Moreover, if B is semisimple then the homomorphism θ is continuous.

Proof. Let M be an arbitrary maximal left ideal of B, then the inverse image $\theta^{-1}(M)$ is a maximal left ideal of A by lemma 1. Thus both difference spaces $A-\theta^{-1}(M)$ and $B-M$ are Banach spaces with the quotient norms. Let $\phi : A \to A-\theta^{-1}(M)$ and ϕ' : $B \to B-M$ be the quotient maps respectively and define a map θ' : $A-\theta^{-1}(M) \to B-M$ by

$$\theta'(\phi(a))=\phi'(\theta(a)) \text{ for each } a \in A.$$

Then, clearly θ' is a well-defined linear map with dense range. If $\theta(\phi(a))=0$ for some a in A, then $\phi'(\theta(a))=0$, hence $\theta(a)$ belongs to M, whence $a \in \theta^{-1}(M)$. Hence we have $\phi(a)=0$. Thus θ' is a linear injection of $A-\theta^{-1}(M)$ into $B-M$. Now we claim that θ' is continuous. Let $\|,\|$ and $|||,|||$ denote the norms on A and B respectively and for simplicity we use the same symbols for the quotient norms on $A-\theta^{-1}(M)$ and $B-M$. Define a second norm $\|,\|_1$ on $A-\theta^{-1}(M)$ by

$$\|\phi(a)\|_1 = |||\theta'(\phi(a))||| \text{ for each } a \in A$$

then $\|,\|_1$ is indeed a norm on $A-\theta^{-1}(M)$ since θ is an injection. Let X denote the regular left A-module $A-\theta^{-1}(M)$ and let π be the left regular representation of A on X. Then π is irreducible by Lemma 2. For simplicity let x' denote the coset $\phi(x)$ for each $x \in A$. Then

$$\pi(a)x' = ax' = (ax)' \text{ for each } a, x \in A.$$

Thus for each $y \in x'$ we have

$$\begin{aligned}\|\pi(a)x'\|_1 &= |||\theta'(ax')||| = |||\phi'(\theta(ay))||| \\ &\leq |||\theta(ay)||| \leq |||\theta(a)||| \cdot |||\theta(y)|||.\end{aligned}$$

The set $\{\theta(y) : y \in x'\}$ is dense in $\theta'(x')$ since $\theta(A) \cap M$ is dense in M, hence

$$\begin{aligned}\|\pi(a)x'\|_1 &\leq |||\theta(a)||| \inf\{|||\theta(y)||| : y \in x'\} \\ &= |||\theta(a)||| \inf\{|||\theta(y)||| : \theta(y) \in \theta'(x')\} \\ &= |||\theta(a)||| \cdot |||\theta'(x')||| \\ &= |||\theta(a)||| \cdot \|x'\|_1.\end{aligned}$$

Therefore, for each a in A $\pi(a)$ is a continuous linear operator on the normed linear space $(X, \|,\|_1)$, and hence the representation π is continuous on A by Johnson's theorem. Hence there is a constant $k>0$ such that

$$\|\pi(a)x'\|_1 \leq k\|a\| \|x'\|_1$$

for each a in A and x' in X. If we denote the identity element of A by e, then

$$\pi(a)e' = (ae)' = a'$$

for each $a \in A$. Now for each $x' \in X$ and each $a \in x'$

$$\|x'\|_1 = \|a'\|_1 = \|\pi(a)e'\|_1 \leq k\|a\|\|e'\|_1.$$

Thus we have

$$\|x'\|_1 \leq k\|e'\|_1 \inf\{\|a\| : a \in x'\} = k\|e'\|_1\|x'\|.$$

From this inequality we see that the identity map from $(X, \|,\|)$ to $(X, \|,\|_1)$ is continuous. Hence the map

$$\theta : (X, \|,\|) \to (B-M, \|\|, \|\|)$$

is continuous.

Now, let $s \in \mathscr{S}(\theta)$. Then there is a sequence $\{a_n\}$ in A such that a_n converges to 0 in A and $\theta(a_n)$ converges to s in B. By the continuity of quotient maps we have

$$\phi(a_n) \to 0 \text{ and } \phi'(\theta(a_n)) \to \phi'(s) \text{ as } n \to \infty.$$

But the composite map $\theta' \circ \phi$ is continuous and $\phi'(\theta(a_n)) = \theta'(\phi(a_n))$ by definition, so $\phi'(s) = 0$. Hence $s \in M$. Since M is an arbitrary maximal left ideal of B and B has the identity, s belongs to $\text{rad}(B)$, the radical of B. Thus we have shown $\mathscr{S}(\theta) \subset \text{rad}(B)$. If B is semisimple, then $\mathscr{S}(\theta) \subset \text{rad}(B) = \{0\}$ and θ is continuous.

References

1. J.M. Bachar et al., *Radical Banach algebras and automatic continuity*, Proceedings Long Beach, Lecture Note 975, Springer-Verlag, Berlin, Heidelberg, 1983.
2. F.F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer-Verlag, NY, 1973.
3. Tae Geun Cho and Jae Chul Rho, *Epimorphisms of noncommutative C*-algebras*, J. Korean Math. Soc. (2) **23**(1986), 185-190.
4. H.G. Dales, *Automatic continuity: A survey*, Bull. London Math. Soc., **10** (1978), 129-183.
5. B.E. Johnson, *The uniqueness of the (complete) norm topology*, Bull. Amer. Math. Soc., **73**(1967), 537-539.

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