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MEROMORPHIC FUNCTIONS OF ORDER ZERO THAT SHARE FOUR VALUES

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1. Introduction

Let $f(Z)$ and $g(Z)$ be meromorphic functions defined on the complex plane. One says that two meromorphic functions f and g share (have the same value) a if $f(z)=a$ if and only if $g(z)=a$.

A theorem of R. Nevalina (see [4]) says that two nonconstant meromorphic functions f and g share five values, then f and g are identical. He also proved that if two distinct meromorphic functions f and g share four values with counting multiplicity, then f is a Möbius transformation of g.

In [1], W. W. Adams and E. G. Straus proved that two polynomials are identical if they share two finite values and two rational functions are identical if they share three finite values.

On the other hand, in [3], G. G. Gundersen proved that if two non-constant meromorphic functions f and g share four values a, b, c, d and a and b both counting multiplicities, then f and g share four values counting multiplicities (if $f \neq g$ then f is a Möbius transformation of g).

In this paper we show that if two nonconstant meromorphic functions of order zero share four values, then they are identical.

The notation and terminology in this paper will follow those in [4]. However the following brief summary may be helpful.

For a meromorphic function f, we write $n(t,f)$ for the number of poles of $f(z)$ in $|z|\leq t$, and

$$N(r,f)=\int_0^r \frac{n(t,f)-n(0,f)}{t}dt+n(0,a)\log r,$$

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$$m(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(re^{i\theta})|d\theta,$$
$$T(r,f)=m(r,f)+N(r,f).$$

Then the Jensen's formula becomes

$$T(r,f)=T\left(r,\frac{1}{f}\right)+0(1).$$

For the simplicity, we write $m(r,a)$, $N(r,a)$, $n(r,a)$ instead of $m\left(r,\frac{1}{f-a}\right)$, $N\left(r,\frac{1}{f-a}\right)$, $n\left(r,\frac{1}{f-a}\right)$ if a is finite, and $m(r,\infty)$, $N(r,\infty)$, $(n(r,\infty)$ instead of $m(r,f)$, $N(r,f)$, $n(r,f)$. Then the First Fundamental Theorem of Nevalinna can be written as

$$m(r,a)+N(r,a)=T(r,f)+0(1)$$

for every a, finite or infinite.

We denote by $\overline{N}(t,a)$ the number of distinct roots of $f(z)-n$ in $|z|\leq t$, and define

$$\overline{N}(r,a)=\overline{N}(r,f,a)=\int_0^r\frac{\overline{N}(t,a)-\overline{N}(0,a)}{t}dt+\overline{N}(0,a)\log r.$$

2. Main results

We will use the Nevalina's Second Fundamental Theorem in the following from;

Second Fundamental Theorem. *Let $f(z)$ be a meromorphic function of order zero, and $a_1, a_2, \cdots, a_q$, where $q>2$, be distinct finite complex numbers. Then*

(1) $((q-1)+0(1))T(r,f)\leq\sum_{\nu=1}^{q}N(r,a_\nu)+N(r,\infty)-N(r,f',0),$

(2) $((q-2)+0(1))T(r,f)\leq\sum_{\nu=1}^{q}\overline{N}(r,a_\nu)-N_0(r,f',0),$

where $N_0(r,f',0)$ counts in N only those zero's of f' which occur other than roots of the equation $f(z)=a_\nu$ ($\nu=1$ to q).

Since $N_0(r,f',0)\geq0$, it is clear from (2) that

(3) $(2+0(1))T(r,f)\leq\sum_{\nu=1}^{4}\overline{N}(r,a_\nu).$

We will use the following results which are proven in [2]. But for the convinience of readers we give a proof.

Lemma 1. *Let f and g be two nonconstant meromorphic functions* *of order zero and share four values* $\{a_1, a_2, a_3, a_4\}$, *then the following conditions holds*:

(4) $$\lim_{r\to\infty}\frac{T(r,f)}{T(r,g)}=1,$$

(5) $$\lim_{r\to\infty}\frac{\sum_{i=1}^{4}\overline{N}(r,a_i)}{T(r,f)}=\lim_{r\to\infty}\frac{\sum_{i=1}^{4}\overline{N}(r,a_i)}{T(r,g)}=2.$$

Proof. One can assume that ∞ is not shared value, because if it is, then we can consider

$$\frac{1}{f-c} \text{ and } \frac{1}{g-c}$$

where c is not a shared value. By (3)

$$\begin{aligned}\sum_{i=1}^{4}\overline{N}(r,a_i)&\leq\overline{N}(r,f-g,0)\leq T(r,f-g,0)\\&\leq T(r,f)+T(r,g)+0(1)\\&\leq(1+0(1))\sum_{i=1}^{4}\overline{N}(r,a_i).\end{aligned}$$

Which means

(6) $$\sum_{i=1}^{4}N(r,a_i)\leq T(r,f)+T(r,g)\leq(1+0(1))\sum_{i=1}^{4}\overline{N}(r,a_i).$$

On the other hand (3) gives

(7) $$\begin{aligned}(2+0(1))T(r,f)&\leq\sum_{i=1}^{4}\overline{N}(r,a_i).\\(2+0(1))T(r,g)&\leq\sum_{i=1}^{4}\overline{N}(r,a_i)\end{aligned}$$

From (6) and (7), we will have (4) and (5). This proves the lemma.

Now we are ready to prove the main results.

THEOREM 1. *Let f, g be two meromorphic functions of order zero so that for four distinct values a_1, a_2, a_3, a_4 we have $f(z)=a_i$ if and only if $g(z)=a_i$; $i=1, 2, 3, 4$. Then f and g are identical.*

Proof. We compare the following two functions

$$G=f'(f-g),\quad F=(f-a_1)(f-a_2)(f-a_3)(f-a_4).$$

It is clear that if F has a zero of order k at z_ν, then G has a zero of order (at least) k at z_ν. Unless $f\equiv g$, we have

(8) $$N(G,r,0)\geq N(F,r,0).$$

On the other hand

$$N(F, r, 0) = \sum_{i=1}^{4} N(r, f-a_i, 0)$$
$$= \sum_{i=1}^{4} N(r, a_i),$$
$$N(G, r, 0) \leq N(r, f', 0) + T(r, f) + T(r, g).$$

Since f is of order zero, it can have at most one deficient value. If f has no deficiency at a_i $(i=1, 2, 3, 4)$, then

$$N(r, F, 0) \geq (4+0(1))\, T(r, f)$$

and

$$N(r, G, 0) \leq 3T(r, f).$$

This contradicts to (8), and we have $f \equiv g$.

Now, suppose that f has a positive deficiency $\delta(1 \geq \delta > 0)$ at a_4. We divide it into two cases;

$$1 > \delta > 0, \text{ and } \delta = 1.$$

First, if $1 > \delta > 0$, then

$$N(r, F, 0) \geq (4 + (1-\delta) + 0(1))\, T(r, f)$$
$$> 3T(r, f) \geq N(r, G, 0).$$

It is impssible, and we have $f \equiv g$.

Finally, let $\delta = 1$. Futher, assume that $a_4 = \infty$. Set f_1, f_2, g_1, and g_2 be four entire functions of order zero such that

$$f = \frac{f_1}{f_2}, \quad g = \frac{g_1}{g_2}$$
$$T(r, f_2) = 0(1)\, T(r, f),$$
$$T(r, g_2) = 0(1)\, T(r, f).$$

Since $N(r, a_4) = 0(1)\, T(r, f)$, we have

$$\lim_{r \to \infty} \frac{\sum_{i=1}^{3} \overline{N}(r, a_r)}{T(r, f)} = 2.$$

Thus, we may assume $a_1 = 0$, and $\overline{N}(r, a_1) \geq \frac{2}{3}$. Let $\{b_1, b_2, \cdots, b_n, \cdots\}$ be the distinct zero points of f, and let

$$h(z) = \prod_{\nu=1}^{\infty} \left(1 - \frac{z}{b_\nu}\right)$$

Then we can find entire functions β and γ such that

$$f_1(z)=h(z)\beta(z),\quad T(r,\beta(z))\leq\frac{1}{3}T(r,f) \text{ as } r\to\infty,$$

$$g_1(z)=h(z)\gamma(z),\quad T(r,\gamma(z))\leq\frac{1}{3}T(r,f) \text{ as } r\to\infty,$$

and

$$(f(z)-g(z))=h(z)\left(\frac{\beta(z)}{f_2(z)}-\frac{\gamma(z)}{g_2(z)}\right).$$

It is clear that

$$\begin{aligned}N(r,f-g,0)&\leq N(r,h(z),0)+T\left(r,\frac{\beta}{f_2}\right)+T\left(r,\frac{\gamma}{g_2}\right)\\&\leq\frac{2}{3}T(r,f)+\frac{1}{3}T(r,f)+\frac{1}{3}T(r,f)+0(1)T(r,f)\\&\leq\frac{4}{3}T(r,f)+0(1)T(r,f).\end{aligned}$$

It follows that

$$\begin{aligned}N(r,G,0)&\leq N(r,f',0)+N(r,f-g)+0(1)T(r,f)\\&\leq\frac{4}{3}T(r,f)+0(1)T(r,f)+N(r,f',0),\end{aligned}$$

and

$$\begin{aligned}N(r,F,0)&\geq\sum_{\nu=1}^{3}N(r,a_\nu)\geq 2T(r,f)+0(1)T(r,f)+N(r,f',0)\\&>\frac{4}{3}T(r,f)+0(1)T(r,f)+N(r,f',0)\\&\geq N(r,G,0).\end{aligned}$$

This is a contradiction, and we have $f\equiv g$. The proof of the theorem is now complete.

Let two entire functions f, g share three finite values a_1, a_2, a_3. Then they have four common values a_1, a_2, a_3, ∞. Hence we have the following corollary.

COROLLARY 1. *Let f and g be two entire functions of order zero and share three finite values. Then they are identical*

It is known that two meromorphic functions f and g share three values, then outside a set of finite measure

$$\lim_{r\to\infty}\sup\frac{T(r,g)}{T(r,f)}\leq 3.$$

For the above result, see [2]. For the entire functions of order zero, we can say a little more;

PROPOSITION 2. *Let f and g be two entire functions of order zero and share two finite values $\{a_1, a_2\}$. Then*

$$\lim_{r\to\infty} \sup \frac{T(r,f)}{T(r,g)} \leq 2.$$

Proof. Set $q=2$ in (1), then

$$(1+0(1))\,T(r,f) \leq \sum_{\nu=1}^{2} N(r,f,a_\nu) + N(r,f,\infty) - N(r,f',0).$$

Since $N(r,f,\infty)=0$, we have

$$\begin{aligned}(1+0(1))\,T(r,f) &\leq \sum_{\nu=1}^{2} N(r,f,a_\nu) - N(r,f',0)\\ &\leq \sum_{\nu=1}^{2} \overline{N}(r,f,a_\nu)\\ &= \sum_{\nu=1}^{2} \overline{N}(r,g,a_\nu)\\ &\leq 2T(r,g).\end{aligned}$$

It follows that

$$\lim_{r\to\infty} \sup \frac{T(r,f)}{T(r,g)} \leq 2.$$

This proves the proposition.

REMARK. There are two different entire functions which share three finite values. The following examples are given in [2].

(i) $f=e^z$, $g=e^{-z}$, shared values $0, \pm 1, \infty$;

(ii) $f=e^{h(z)}$, $g=\frac{1}{2}(e^{h(z)}+a_1^2 e^{-h(z)})$, shared values, $a_1 \neq 0$, $a_2=-a_1$, ∞.

But we have noexample of two different entire functions of order zero which share two finite values.

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