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AMALGAMATION IN CERTAIN SMALL VARIETIES

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Introduction

This paper is concerned with amalgamation classes of residually small varieties. Congruence distributive varieties are considered in section 3. Here we prove the result implicit in Day [5]: if $\mathscr{V}$ is a congruence distributive variety generated by a finite simple algebra then $\mathscr{V}$ satisfies the amalgamation property.

Section four is concerned with amalgamation classes of abelian group varieties. Here it is shown that every proper subvariety of the variety of all abelian groups satisfies the amalgamation property. The concluding section contains some open problems suggested by the results of the paper.

1. Preliminaries

Congruences

The congruence lattice of an algebra A is denoted by Con(A). We use the symbol Δ for the trivial congruence. Given $\theta \in \mathrm{Con}(A)$ we shall say that

(1) θ is non-zero if for some $a, b \in A$ with $a \neq b$ we have $(a, b) \in \theta$.

(2) θ is a B-congruence on A if $A/\theta \cong B$ for some algebra B.

(3) θ is filtral if $A = \prod_{i \in I} A_i$ is a product of algebras and θ is induced by some ultrafilter on I.

Algebras with a unique smallest non-zero congruence are called subdirectly irreducible. The well-known result of G. Birkhoff

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states that every nontrivial algebra is either subdirectly irreducible or has a subdirect decomposition via its subdirectly irreducible images.

Absolute retracts and essential extensions

The symbolism $A \leq B$ (f embeds A into B) indicates that an algebra B is an extension of A. An extension B of an algebra A is said to be essential if each non-zero congruence of B restricts to a non-zero congruence of A. An algebra $A \in \mathscr{K}$ is said to be an absolute retract in $\mathscr{K}$ if, for any embedding $f : A \to B$, there is an epimorphism $g : B \to A$ such that gf is the identity map on A.

The next result can be found in [12].

Proposition 1.1. (1) *An essential extension of a subdirectly irreducible algebra is subdirectly irreducible.*

(2) *If B is an extension of A then among the congruences θ on B with $\theta/A \neq \Delta$ there is a maximal one θ_0 and the extension $A \leq B/\theta_0$ is essential.*

(3) *An algebra has a proper essential extension iff it is not an absolute retract.*

Varieties

A variety of algebras is denoted by $\mathscr{V}$. The following two facts are well-known

(1) $\mathscr{V}$ is closed under the formation of products, subalgebras and homomorphic images

(2) Every member of $\mathscr{V}$ is a subdirect product of subdirectly irreducible members of $\mathscr{V}$. Call $\mathscr{V}$ congruence distributive if, for any $A \in \mathscr{V}$, Con(A) is a distributive lattice.

Theorem 1.2. (Jipsen-Rose [9]). *Let $\mathscr{V}$ be a congruence distributive variety and assume that every member of $\mathscr{V}$ has a one-element subalgebra. Then every direct product of absolute retracts in $\mathscr{V}$ is an absolute retract in $\mathscr{V}$.*

Amalgamation in a variety

By a diagram in a variety $\mathscr{V}$ we mean a quintuple (A, f, B, g, C)

with $A, B, C \in \mathscr{V}$ and $f : A \to B$, $g : A \to C$ embeddings. By an amalgam of this diagram in $\mathscr{V}$ we mean a triple (D, f_1, g_1) with $D \in \mathscr{V}$ and with $f_1 : B \to D$, $g_1 : C \to D$ embeddings such that $f_1 f = g_1 g$. If such an amalgam exists, we say that the diagram can be amalgamated in $\mathscr{V}$. An algebra $A \in \mathscr{V}$ is called an amalgamation base for $\mathscr{V}$ if every diagram (A, f, B, g, C) can be amalgamated in $\mathscr{V}$. The class of all amalgamation bases for $\mathscr{V}$ is called the amalgamation class and denoted by Amal($\mathscr{V}$). The variety $\mathscr{V}$ is said to satisfy the amalgamation property if $\mathscr{V} = \mathrm{Amal}(\mathscr{V})$.

Amalgamation classes of varieties (more generally of elementary classes) were studied by Yasuhara [17]. In particular we have the following

THEOREM 1.3. (1) *Amal*($\mathscr{V}$) *is a proper class*
(2) *The complement of Amal*($\mathscr{V}$) *is closed under ultrapowers*
(3) *Every absolute retract of* $\mathscr{V}$ *belongs to Amal*($\mathscr{V}$).

COROLLARY 1.4. *Amal*($\mathscr{V}$) *is elementary iff it is closed under ultraproducts.*

Proof. A class $\mathscr{K}$ is elementary iff $\mathscr{K}$ is closed under ultraproducts and isomorphisms and the complement of $\mathscr{K}$ is closed under ultrapowers (see [5] Corollary 6.1.16). Theorem 1.3 (2) yields the result.

The following lemma makes the problem of amalgamating a diagram somewhat more accessible.

LEMMA 1.5 (Grätzer and Lakser [8]). *A diagram* (A, f, B, g, C) *in a variety* $\mathscr{V}$ *can be amalgamated iff for all* $u \neq v \in B$ *there exists* $D \in \mathscr{V}$ *and homomorphisms* $f' : B \to D$ *and* $g' : C \to D$ *such that* $f \cdot f' = g' \cdot g$ *and* $f'(u) \neq f'(v)$ *and the same holds for C.*

For our next result we assume that every member of $\mathscr{V}$ has a one-element subalgebra.

Consider a product $A = \prod_{\gamma \in \alpha} A_\gamma$ of members of $\mathscr{V}$. For each $\gamma \in \alpha$ we have an embedding $\bar{\gamma} : A_\gamma \to A$ where for $a_\gamma \in A_\gamma$ and $\beta \neq \gamma$ the β^{th} coordinate of $\bar{\gamma}(a_j)$ is an element $e_\beta \in A_\beta$ with $\{e_\beta\}$ a one-element subalgebra of A_β.

Let $B=\prod_{i\in I} S_i$ be a product of subdirectly irreducible members of $\mathscr{V}$ and assume that $f: A\to B$ is a subdirect decomposition. Then for each $\gamma\in\alpha$ there is a subset I_γ of I and an embedding $f\cdot\bar{\gamma}: A_\gamma\to \prod_{i\in I_\gamma} S_i=B_\gamma\leq B$. For the next lemma we assume that $A_\gamma\in\text{Amal}(\mathscr{V})$ for any $\gamma\in\alpha$.

LEMMA 1.6. *Let $f: A\to B$ and $g: A\to C$ be subdirect decompositions. Then the diagram (A, f, B, g, C) can be amalgamated in $\mathscr{V}$.*

Proof. For $\gamma\in\alpha$ let $(D_\gamma, h_\gamma, K_\gamma)$ be an amalgam of $(A, f\cdot\bar{\gamma}, B_\gamma, g\cdot\bar{\gamma}, C_\gamma)$. Let D, h and K be products of D_γ's, h_γ's and K_γ's. Then (D, h, K) is an amalgam of (A, f, B, g, C).

2. Residually small varieties

A variety $\mathscr{V}$ is said to be residually small if $\mathscr{V}$ satisfies the two equivalent conditions of the following theorem:

THEOREM 2.1. (Taylor [16]). (1) *There exists a cardinal α such that every subdirectly irreducible member of $\mathscr{V}$ has cardinality $\leq\alpha$.*

(2) *Every member of $\mathscr{V}$ has a maximal essential extension in $\mathscr{V}$.*

Combining Proposition 1.1 and Theorem 2.1 we have the following:

COROLLARY 2.2. *Let $\mathscr{V}$ be a residually small variety. Then every subdirectly irreducible member of $\mathscr{V}$ has a maximal essential extension which is subdirectly irreducible. Moreover, every such maximal essential extension is an absolute retract in $\mathscr{V}$ and therefore does not have proper essential extensions.*

We define a maximal irreducible algebra in a variety $\mathscr{V}$ to be a subdirectly irreducible algebra of $\mathscr{V}$ with no essential extensions in $\mathscr{V}$. Let $\mathscr{V}_{MI}$ be the class of all maximal irreducibles in $\mathscr{V}$ and $P(\mathscr{V}_{MI})$ be the class of all products of members of $\mathscr{V}_{MI}$.

COROLLARY 2.3. *Let $\mathscr{V}$ be a residually small variety. Then every* *subdirectly irreducible member of* $\mathscr{V}$ *has a maximal essential extension in* $\mathscr{V}_{MI}$ *and therefore every member of* $\mathscr{V}$ *is embeddable in a member of* $P(\mathscr{V}_{MI})$.

COROLLARY 2.4. *Let* A *be a member of a residually small variety* $\mathscr{V}$. *Then* $A \in Amal(\mathscr{V})$ *if and only if for any two embeddings* $f : A \to B \in P(\mathscr{V}_{MI})$ *and* $g : A \to C \in P(\mathscr{V}_{MI})$ *the diagram* (A, f, B, g, C) *can be amalgamated in* $\mathscr{V}$.

Proof. The condition is clearly necessary. To prove sufficiency, observe that if $f : A \to B' \in \mathscr{V}$ and $g : A \to C \in \mathscr{V}$ are two embeddings then, since $B' \leq B$ and $C \leq C'$ for some $B, C \in P(\mathscr{V}_{MI})$ (see Corollary 2.3), the diagrams (A, f, B', g, C') and (A, f, B, g, C) have the same amalgam in $\mathscr{V}$.

DEFINITION 2.5. (1) An algebra $A \in \mathscr{V}$. is said to be injective in $\mathscr{V}$ if for any embedding $f : B \to C \in \mathscr{V}$ and any homomorphism $g : B \to A$ there is a homomorphism $h : C \to A$ such that $g = h \cdot f$.

(2) Let $\mathscr{V}$ be a residually small variety. An algebra $A \in \mathscr{V}$ is said to satisfy the property (Q) if for any homomorphism $g : A \to M \in \mathscr{V}_{MI}$ and any embedding $f : A \to B \in \mathscr{V}$ there is a homomorphism $h : B \to M$ such that $g = h \cdot f$.

The following result is crucial in our investigations.

LEMMA 2.6 (Bergman [1]). *Let* $\mathscr{V}$ *be a residually small variety and* $A \in \mathscr{V}$. *If* A *satisfies the property* (Q) *then* $A \in Amal(\mathscr{V})$.

COROLLARY 2.7. *Let* $\mathscr{V}$ *be a residually small variety and assume that every member of* $\mathscr{V}_{MI}$ *is injective in* $\mathscr{V}$. *Then* $\mathscr{V}$ *satisfies the amalgamation property.*

For the rest of this section we assume that $\mathscr{V}$ is a residually small variety satisfying the following three conditions:

(*) Every member of $\mathscr{V}$ has a one-element subalgebra

(**) If $A \in \mathrm{Amal}(\mathscr{V})$ then A is a subdirect product of subdirectly irreducible members of $\mathrm{Amal}(\mathscr{V})$

(***) If B is a product of subdirectly irreducible members of $\mathrm{Amal}(\mathscr{V})$ then $B \in \mathrm{Amal}(\mathscr{V})$.

REMARK 2. 8.

(α) It follows from (***), Theorem 1. 3 (3) and Corollary 2. 2 that $P(\mathscr{V}_{MI}) \subset \mathrm{Amal}(\mathscr{V})$.

(β) By (*) every factor of a product is embeddable in a product.

LEMMA 2. 9. *Let A be a subdirect product of subdirectly irreducible members of $Amal(\mathscr{V})$. Then $A \in Amal(\mathscr{V})$ if and only if any diagram (A, b, B, c, C) can be amalgamated in $\mathscr{V}$, where $b: A \rightarrow B$ and $c: A \rightarrow C$ are subdirect decompositions of A.*

Proof. Clearly the condition is necessary. We shall use Corollary 2. 3 to prove the sufficiency. Let $d: A \rightarrow D$ and $e: A \rightarrow E$ be embeddings where $D, E \in P(\mathscr{V}_{MI})$. The embeddings d and e induce subdirect decompositions $b: A \rightarrow B$ and $c: A \rightarrow C$ (see figure 1). It follows from Lemma 1. 5 that there are embeddings $d': B \rightarrow D$ and $e': C \rightarrow E$ such that $d' \cdot b = d$ and $e' \cdot c = e$. By our assumption (A, b, B, c, C) has an amalgam (F, c', b'). By Corollary 2. 3 we may assume that $F \in P(\mathscr{V}_{MI})$. It follows that $F \in \mathrm{Amal}(\mathscr{V})$ by the Remark 2. 8. The assumption of our lemma and Corollary 2. 4 imply that the diagrams (B, b', F, d', D) and (C, e', E, c, F) have amalgams (G, f, g) and (H, h, s) in $\mathscr{V}$. Since $F \in \mathrm{Amal}(\mathscr{V})$ the

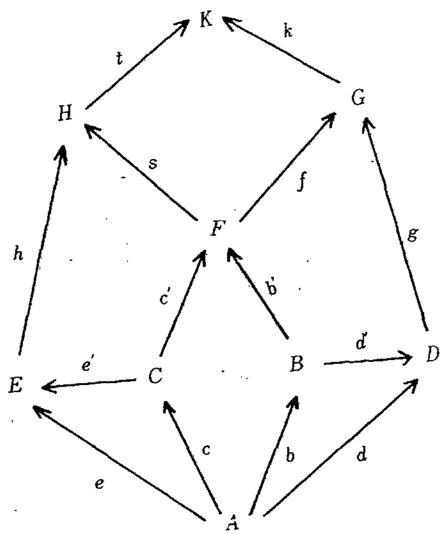

FIGURE 1.

diagram (F, f, G, s, H) has an amalgam (K, t, k) and so (K, kg, th) is an amalgam of (A, d, D, e, E).

THEOREM 2.10. *Amal($\mathscr{V}$) is closed under direct products.*

Proof. Follows directly from Lemmas 1.6 and 2.9.

EXAMPLE 2.11. The "smallest non-modular lattice", the so-called pentagon, is pictured below in figure 2. Let $\mathscr{V}$ be a variety generated by the pentagon. It follows from a result of C. Bergman ([11]) that every member of Amal($\mathscr{V}$) is a subdirect power of the pentagon. Since the pentagon is the only maximal subdirectly irreducible member of $\mathscr{V}$ it follows from Theorem 2.10 that Amal($\mathscr{V}$) is closed under direct products.

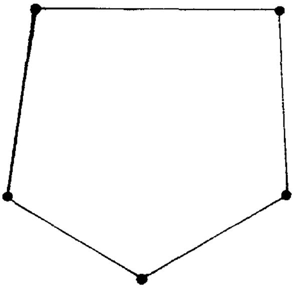

FIGURE 2.

3. Congruence distributive varieties

All algebras and varieties in this section are assumed to be congruence distributive.

The following result of Jónsson is fundamental for congruence distributive algebras and varieties.

THEOREM 3.1 (Jónsson [10]). (1) *Let $A = \Pi A_i$ be a product of algebras and $B \leq A$. If $\theta \in Con(B)$ is such that B/θ is subdirectly irreducible then there is a filtral congruence φ on A such that $\varphi/B \subset \theta$.*

(2) *If $\mathscr{V}$ is a variety generated by a finite algebra A then every subdirectly irreducible member of $\mathscr{V}$ is an image of a*

subalgebra of A.

COROLLARY 3.2. *Let K^I be a direct power of a finite subdirectly irreducible algebra K. Then φ is a K-congruence on K^I if and only if φ is filtral.*

Proof. If φ is filtral then $K^I/\varphi \cong K$ (see for instance [4] Corollary 4.2.5). Since K is a finite algebra the converse follows from Theorem 3.1 (1).

COROLLARY 3.3. *Let $\mathscr{V}$ be a variety generated by a fnite subdirectly irreducible algebra K and assume that every subdirectly irreducible member of $\mathscr{V}$ is embeddable in K. If $A \leq B \in \mathscr{V}$ then every K-congruence on A can be extended to a K-congruence on B.*

Proof. Let $B \to \Pi S_i$ be a subdirect decomposition. By assumption S_i can be embedded in K for each $i \in I$ so that $A \leq B \leq K^I$. If $\theta \in \mathrm{Con}(A)$ with $A/\theta \cong K$ then, since K is subdirectly irreducible and finite, it follows from Theorem 3.1 (1) that $\varphi/A = \theta$ for some filtral congruence φ on K^I. By Corollary 3.2 φ is a K-congruence on K^I so that φ/B is a K-congruence on B and so it is a K-congruence on A.

We are now ready to prove the following

THEOREM 3.4 (c.f. Day [4]). *Let $\mathscr{V}$ be a congruence distributive variety generated by a fınite subdirectly irreducible algebra K. Suppose further that K has no non-trivial subalgebras. Then K is injective in $\mathscr{V}$ so that every member of $\mathscr{V}$ satisfies the property (Q) and therefore $\mathscr{V}$ satisfies the amalgamation property.*

Proof. It follows from Theorem 3.1 (2) that K is the only subdirectly irreducible member of $\mathscr{V}$. The result follows from Corollaries 3.3 and 2.7.

EXAMPLE 3.5. The following two facts about lattices are well-known: (1) Lattices are congruence distributive algebras and (2) up to isomorphism the two-element chain is the only subdirectly irreducible distributive lattice.

Thus by Theorem 3.4 the variety of all distributive lattices satisfies the amalgamation property. This was first proved by Pierce [15].

4. Varieties of abelian groups

Call a variety $\mathscr{V}$ of groups abelian if every member of $\mathscr{V}$ is an abelian group. Note that, if G is an abelian group, $\mathrm{Con}(G)$ is the subgroup lattice of G. The following facts are well-known (Fuchs [6]).

(1) Every subdirectly irreducible abelian group G is isomorphic to Z_{p^k} where p is a prime number and $k \subset Z^+ U\{\infty\}$. Thus every abelian group variety is residually small.

(2) If $\mathscr{V}$ is a proper subvariety of the variety of all abelian groups then $\mathscr{V}$ is determined by the identity $nx=0$ (and the abelian identity $x+y=y+x$).

Using 1 and 2 we have:

(3) If $\mathscr{V}$ is the variety of all abelian groups then $\mathscr{V}_{MI}=\{Z_{p^\infty} : p$ is a prime number$\}$.

(4) If $\mathscr{V}$ is a proper subvariety of the variety of all abelian groups, that is $\mathscr{V}$ is determined by the identity $nx=0$ where n has prime decomposition $n=p_1^{k_1}\cdots p_m^{k_m}$, then the subdirectly irreducible members of $\mathscr{V}$ are $Z_{p_i^{l_i}}$ for $i \in \{1, \cdots, m\}$ and $l_i \leq k_i$. It follows that $\mathscr{V}_{MI}=\{Z_{p_1^{k_1}}, \cdots, Z_{p_m^{k_m}}\}$.

In this section we shall prove the following:

THEOREM 4.1. *Let $\mathscr{V}$ be an abelian group variety. Then every $\mathscr{V}_{MI}$ is injective in $\mathscr{V}$.*

COROLLARY 4.2. *Every member of $\mathscr{V}$ satisfies the property (Q) and therefore $\mathscr{V}$ satisfies the amalgamation property.*

REMARK 4.2. It is well-known that if $\mathscr{V}$ is the variety of all abelian groups then $\mathscr{V}$ satisfies the amalgmation property. In Neumann [13] this is shown by defining a "generalized free product". If groups A and B with amalgamated subgroup C are to satisfy the amalgamation property, Neumann examines what conditions on A, B, and C allow the existence of the generalized free product into which A and B can be embedded so as to have

C as their amalgamated subgroup. It is shown that if A, B and C are all abelian then there always exists an abelian group P^+ into which A and B can be embedded so as to have amalgamated subgroup C.

It is also known that Z_{p^∞} is injective in the variety of all abelian groups for each prime number p.

For our next lemma we assume that $\mathscr{V}$ is an abelian variety determined by the identity $p^k \cdot x=0$ where p is a prime number and k is a positive integer. For an embedding $f: A \rightarrow B \in \mathscr{V}$ and $b \in (B-f(A))$ we let A^* be the subgroup of B generated by the set $f(A) \cup \{b\}$.

LEMMA 4.3. *Let $g: A \rightarrow Z_{p^n}$ be a homomorphism. Then there is a homomorphism $h: A^* \rightarrow Z_{p^n}$ such that $g=h \cdot f$.*

Proof. To simplify the notation we identify A with a subgroup $f(A)$ of B. Let $r \in Z^+$ be such that $rb \in A$. We have to show that, if $g(rb)=y$, then there exists $z \in Z_{p^n}$ such that $rz=y$.

The elements rb and y have order $p^n/(p^n, r)$ and $p^n/(p^n, y)$ respectively. Thus there exists an integer k such that $k(p^n, r)=(p^n, y)$. Also there are relatively prime integers u and v with $r=r^*u$ and $p^n=r^*v$ and integers s and t such that $rsk+tkp^n=(p^n, y)$. If $y=(p^n, y)k^*$ for an integer k^* then $y \equiv rskk^* (\text{mod } p^n)$. Put $z=skk^*$ to obtain the desired element of Z_{p^n}.

For the next lemma we assume that $\mathscr{V}$ is an abelian variety determined by the identity $nx=0$. Let $n=p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ be a decomposition of n in terms of distinct prime numbers so that $Z_{p_1^{\alpha_1}}, \cdots, Z_{p_k^{\alpha_k}}$ are the only members of $\mathscr{V}_{MI}$. Let $f: A \rightarrow B \in \mathscr{V}$ be an embedding and $g: A \rightarrow X$ a homomorphism, where X is a finite product of members of $\overline{\mathscr{V}_{MI}}$. As before, for $x \in (B-f(A))$ we let A^* be the subgroup generated by $f(A) \cup \{X\}$.

LEMMA 4.4. *There is a homomorphism $h: A^* \rightarrow X$ such that $g=h \cdot f$.*

Proof. As before we identify A and $f(A)$. Since $nx=0$ there exist $x_1, x_2, \cdots, x_k$ with $x_i \in Z_{p_i^{\alpha_i}}$, $x_i=\beta_i \cdot x$ for some β_i and $x=x_1+x_2$

$+\cdots+x_k$. Supposing f defined on rx implies therefore that f is defined on each rx_i. If p_i is not a prime dividing the order of the group generated by x then $f(x_i)=0$ will do. If p_i does divide this order then, by Lemma 4.3, we may find $z_i \in Z_{p_i^{n_i}}$ which extends f to x_i. The f so obtained is a homomorphism.

Corollary 4.5. *Let $\mathscr{V}$ and X be as in Lemma 4.4. Then X is injective in $\mathscr{V}$.*

Proof. Let $f: A \to B \in \mathscr{V}$ be an embedding and $g: A \to X$ a homomorphism. For $b \in B$ let $\langle b \rangle$ be the cyclic group generated by b. Well-order the set $\{\langle b \rangle : b \in (B-f(A))\}$. Apply induction, using Lemma 4.4 on successor stages and taking the union of increasing chains of subgroups and homomorphisms on limit stages.

Proof of Theorem 4.1. The second statement of the theorem follows from the first (see Corollary 2.7). If $\mathscr{V}$ is the variety of all abelian groups then for any prime p, the group Z_{p^∞} is injective. If $\mathscr{V}$ is a proper subvariety of the variety of all abelian groups the result follows from Corollary 4.5.

5. Some open problems

5.1 For which non congruence-distributive varieties does Theorem 1.2 hold. Varieties of groups would be a possibility: although groups are not congruence distributive algebras, they do have one-element subalgebras.

5.2 It is shown in Jipsen & Rose [9] that if $\mathscr{V}$ is a residually small congruence distributive variety whose members have a one-element subalgebra then the members of Amal($\mathscr{V}$) are the algebras which satisfy the property (Q). Thus, for this variety the converse of Lemma 2.6 holds. Can one generalize this result for other varieties?

5.3 By Theorem 4.1, for any natural number n, the abelian group variety determined by the identity $rx=0$ satisfies the amalgamation property. On the other hand it is shown in Neumann [14] that, for $n=4$, the amalgam of two groups of exponent 4 cannot have exponent 4. Thus for $n=4$ the variety of groups determined by the identity $x^n=e$ fails to satisfy the amalgamation property. For which n does it satisfy this property? Can one characterize the amalgamation class of such varieties? Are they residually small?

5.4 It will be interesting to find the necessary and sufficient conditions for amalgamation classes of varieties of particular algebras (e.g. groups or lattices) to be elementary. The residually small varieties are probably the most accessible for this sort of problem. For instance, if $\mathscr{V}$ is a variety from Theorem 2.1 and Amal($\mathscr{V}$) is closed under homomorphic images, then, by Corollary 1.4, Amal($\mathscr{V}$) is elementary. Examples of varieties whose amalgamation class is (not) elementary can be found in Bergman [1] ([3]).

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