J. Korean Math. Soc. 26(1989), No. 1, pp. 67~74

CLOSED SEMI-IDEALS IN A II_1-FACTOR

SA GE LEE, SUNG JE CHO and SEUNG-HYEOK KYE

1. Introduction

Throughout, M will be a fixed II_1-factor with normalized trace τ. A (nonempty) subset S of M is called a semi-ideal in M if $xSy \subset S$ for all $x, y \in M$.

In Section 2, we determine the class of all (norm) closed semi-ideals in M. The height $h(S)$ of a semi-ideals in M is defined by

$$h(S) = \sup\{\tau(p) : p \in P(S)\},$$

where $P(\cdot)$ denotes the set of all projections in the set $(\cdot)$, through this work. We say that $h(S)$ is accessible if there is $p \in P(S)$ such that $\tau(p) = h(S)$. Otherwise, it is called inaccessible (cf. [7] Definition 2).

In Section 3, we describe the spectrum $\sigma_t(X)$ of $x \in M$, $0 < t \leqq 1$, modulo the closed semi-ideal J_t, where J_t is uniquely determined as the closed semi-ideal in M whose height $h(J_t)$ is inaccessible and $h(J_t) = t$.

2. The closed semi-ideals

For every $t \in (0, 1]$, we put

$$I_t = \{x \in M : \tau(l(x)) < t\} \text{ and}$$

$$J_t = I_t^-, \text{ the norm closure of } I_t,$$

where $l(x)$ denotes the left support (projection) of $x \in M$. It is immediate to verify that I_t, J_t are semi-ideals in M.

In what follows, H will be the underlying Hilbert space on which operators of M act, and $P(M)$ will be abbreviated by P.

PROPOSITION 2.1. *Let $x \in M$, $0 < t < 1$. The following conditions are mutually equivalet.*

Received May 30, 1988.
Supported by a grant from KOSEF, 1987.

(i) $x \in J_t$.

(ii) *If* $q \in P$ *and* $q(H) \subset x(H)$, *then* $\tau(q) < t$.

(iii) *If* x *is bounded below on* $p(H)$ *for* $p \in P$, *then* $\tau(p) < t$.

(iv) *For every* $\varepsilon > 0$, *there is* $p \in P$ *such that* $\|xp\| < \varepsilon$ *and* $\tau(1-p) < t$.

Proof. (i)→(ii). If we modify ([1] Theorem 1) slightly, we see that $q = xy$ for some $y \in M$. Then $q \in J_t$, since J_t is a semi-ideal in M. Let $\{q_n\}$ be a sequence in I_t such that $\|q - q_n\| \to 0$. One can easily check that the operator $q_n | q(H) : q(H) \to H$ is bounded below for all sufficiently large n's. Let us fix one such n. Then $q_n q$ has the kernel $(I-q)(H)$ and has the closed range. Note that $l(q_n q) \sim r(q_n q)$ in M, where $r(\cdot)$ denotes the right support of the element $(\cdot)$, and that $r(q_n q) = q$. Consequently, $\tau(q) = \tau(r(q_n q)) = \tau(l(q_n q)) \leqq \tau(q_n) < t$, as desired.

(ii)→(iii). Let $p \in P$ be given such that x is bounded below on $p(H)$. We put $q = l(xp) \in P$. Then $q(H) \subset x(H)$. By hypothesis (ii), $\tau(q) < t$. On the other hand, $r(xp) = p$, since $\ker(xp) = \ker p$. It follows that $\tau(p) = \tau(r(xp)) = \tau(l(xp)) = \tau(q) < t$.

(iii)→(iv). Let $E(\cdot)$ be the spectral measure of $|x|$. If we put $p = E[0, \varepsilon/2)$, then it is easily seen that p is a required one in (iv), by a semilar argument as in the proof of ([5] Lemma 2.5).

(iv)→(i). Let $\varepsilon (>0)$ be given arbitrary. By (iv), there is $p \in P$ such that $\|xp\| < \varepsilon$ and $I - p \in I_t$. Note that $x(I-p) \in I_t$. Now $\|x - x(I-p)\| = \|xp\| < \varepsilon$. Hence $x \in \bar{I_t} = J_t$.

COROLLARY 2.2. *For every semi-ideal* J_t, *the height* $h(J_t)$ *is inaccessible.*

Proof. Clearly $h(Jt) = h(I_t) = t$. Assume contrary that $\tau(p) = t$ for some $p \in P(J_t)$. By (i)→(ii) of Proposition 2.1, $\tau(p) < t$, which is a contradiction.

LEMMA 2.3. *A (norm) closed semi-ideal* S *of* M *is determined by the projections contained in* S.

Proof. We have to show the following: If $S_i (i=1,2)$ are two closed semi-ideals in M such that $P(S_1) = P(S_2)$, then $S_1 = S_2$.

Let $x \in S_1$. Let $x = u|x|$ be the left polar decomposition of x and $E(\cdot)$ be the spectral measure of $|x|$. If we put $e_n = E[1/n, \infty)$, $n = 1, 2, \cdots$, then

$$(*) \quad \| |x| - |x| e_n \| = \| |x| E[0, 1/n) \| \leqq 1/n \to 0.$$

Here $|x| e_n \in S_1$, since $|x| = u^* x \in S_1$. Note that $|x| e_n$ has the closed range, so that $l(|x| e_n)(H) = (|x| e_n)(H) \subset |x|(H)$. By the modified version of the Douglas result ([1] Theorem 1) mentioned already, $l(|x| e_n) = |x| y$ for some $y \in M$. Hence $l(|x| e_n) \in S_1$. Now $l(|x| e_n) \sim r(|x| e_n) = e_n$, which implies that $e_n \in P(S_1)$, since two equivalent projections in a finite von Neumann algebra are unitarily equivalent. As we have assumed that $P(S_1) = P(S_2)$, we have $e_n \in P(S_2)$ and hence $|x| e_n \in S_2$. By (*), $|x| \in S_2$ and consequently $x \in S_2$. We have shown that $S_1 \subset S_2$. The reverse inclusion is proven by the symmetric argument.

LEMMA 2.4. *If J is a (norm) closed nontrivial ($\{0\} \subsetneqq J \subsetneqq M$) semi-ideal of M such that $h(J)$ is inaccessible, then there is a unique $t \in (0, 1]$ such that $J = J_t$.*

Proof. To see the uniqueness, let $0 < t_1 < t_2 \leqq 1$. Find $p \in P$ such that $\tau(p) = t_1$. Then $p \notin J_{t_1}$ by Corollary 2.2, while $p \in J_{t_2}$. Hence $J_{t_1} \subsetneqq J_{t_2}$.

Now let J be a closed semi-ideal of M such that $\{0\} \subsetneqq J \subsetneqq M$ and $h(J)$ is inaccessible. Put $t = h(J)$. Since $P(J) \neq \{0\}$ (Lemma 2.3), we see that $t \in (0, 1]$. By inaccessibility of $h(J)$, $P(J) \subset P(I_t) \subset P(J_t)$, which, in turn, implies that $J \subset J_t$ (Lemma 2.3). To get $J_t \subset J$, let $p \in P(J_t)$. Then $\tau(p) < t$, as we saw already. By definition of t, $\tau(p) < t(q)$ for some $q \in P(J)$. Thus, $p \sim q_1 \leqq q$ for some $q_1 \in P$. Since $q_1 = q_1 q \in J$, we have that $p \in J$. Hence $P(J_t) \subset P(J)$, and consequently $J_t \subset J$ (Lemma 2.3).

For every $t \in [0, 1]$, let us define

$$K_t = \{x \in M : \tau(l(x)) \leqq t\}.$$

One can easily show that K_t is a norm closed semi-ideal of M and that $h(K_t)$ is accessible. The converse holds as in the following lemma. We omit the proof, as it is dealt with the similar way as the case of J_t's.

LEMMA 2.5. *K is a closed semi-ideal of M whose height $h(K)$ is accessible if and only if there is a unique $t\in[0,1]$ such that $K=K_t$.*

In ([3] Definition 2.1), the t-th singular number of τ-measurable operator T is defined by

$$\mu_t(T)=\inf\{\|Tp\| : p\in P \text{ and } \tau(1-p)\leqq t\},$$

where $t\in(0,\infty)$.
When $x\in M$, $t\in[0,1]$, Proposition 2.4 of [3] implies that

$$\mu_t(x)=\text{dist}(x, K_t),$$

where dist denotes the distance.
For $x\in M$, $t\in(0,\infty)$, let us define

$$\nu_t(x)=\inf\{\|xp\| : p\in P \text{ and } \tau(1-p)<t\}.$$

The next two propositions are analogues of Proposition 2.2 and Proposition 2.4 in [3], respectively. We shall omit their proofs which go parallel to the corresponding ones in [3].

PROPOSITION 2.6. *For $x\in X$, $t\in(0,\infty)$, we have*

$$\nu_t(x)=\inf\{s\geq 0 : \lambda_s(x)<t\},$$

where $\lambda_s(x)=\tau(E(s,\infty))$ and $E(\cdot)$ is the spectral measure for $|x|$.

PROPOSITION 2.7. *For $x\in M$, $t\in(0,1]$, we have*

$$\nu_t(x)=dist(x, J_t).$$

REMARK 2.8. For every $t\in(0,\infty)$, $x\in M$, it is clear that $\mu_t(x)\leqq\nu_t(x)$. When $t\in(0,1]$, $\nu_t(x)$ is right continuous at t if and only if $\nu_t(x)=\mu_t(x)$. Because of this fact and similarity between definitions of $\mu_t(x)$ and $\nu_t(x)$, many assertions in [3], for example, Lemma 2.5 and Proposition 2.7 there, can be formulated in terms of $\mu_t(x)$.

3. Invertibility modulo J_t

Let S be a closed semi-ideal of M and $x\in M$. We say that x is left invertible in M modulo S if there is $y\in M$ such that $yx-I\in S$. An element $x\in M$ is called invertible in M modulo S if there is $y\in M$ such that $yx-I\in S$ and $xy-I\in S$.

If K is a closed subspace of H and p is the projection onto K such that $p\in M$, we shall also write $K\in M$ and $\tau(K)$ to mean

$\tau(p)$. The next two lemmas are von Neumann algebra version of Lemma 1.1 and 1.2 in [2], respectively. To prove Lemma B, one has to apply the parallelogram law for projections in M. We omit the obvious proofs.

LEMMA A. *Let $x\in M$. For every $\varepsilon>0$, there is a closed subspace K of H such that kernel$(x)\subset K$, $K\in M$,*

$$\|x\xi\|<\varepsilon\|\xi\|, \text{ for all } \xi\in K \text{ and}$$
$$\|x\xi\|\geqq\varepsilon\|\xi\|, \text{ for all } \xi\in K^{\perp}.$$

(When $K=\{0\}$, the first inequality is vacuous.)

LEMMA B. *Let $x\in M$. For $\varepsilon>0$, suppose that K is a closed subspace of H such that $K\in M$, $\|x\xi\|<\varepsilon\|\xi\|$ for all $\xi\in K$ with $\xi\neq 0$, and that L is a closed subspace of H such that $L\in M$, $\|x\xi\|\geqq\varepsilon\|\xi\|$ for all $\xi\in L^{\perp}$. Then*

$$\tau(K)\leqq\tau(L),$$
$$\tau(L^{\perp})\leqq\tau(K^{\perp}).$$

PROPOSITION 3.1. *For $x\in M$, $t\in(0,1]$, the following conditions are mutually equivalent.*

(i) *x is left invertible modulo I_t.*
(ii) *x is left invertible modulo J_t.*
(iii) *x is bounded below on $p(H)$ for some $p\in P$ with $\tau(1-p)<t$.*
(iv) *The nullity $\nu(x)<t$, where $\nu(x)=\tau(\text{kernel}(x))$.*
(v) *x is intertible modulo I_t.*
(vi) *x is invertible modulo J_t.*

Proof. (i)→(ii). Trivial, since $I_t\subset J_t$.

(ii)→(iii). Assume that $yx-I\in J_t$ for some $y\in M$. Note that $y\neq 0$. By Lemma A, there is a closed subspace K of H such that $K\in M$,

$$\|yx\xi\|<(1/2)\|\xi\| \text{ for } \xi\in K \text{ with } \xi\neq 0$$

and

$$\|yx\xi\|\geqq(1/2)\|\xi\| \text{ for all } \xi\in K^{\perp}.$$

Thus $\|x\xi\|\geqq(1/(2\|y\|))\|\xi\|$, for all $\xi\in K^{\perp}$. It suffices to show that $\tau(K)<t$. For all $\xi\in K$, we have

$$\|(I-yx)\xi\|\geqq\|\xi\|-\|yx\xi\|\geqq\|\xi\|-(1/2)\|\xi\|=(1/2)\|\xi\|,$$

which shows that I-yx is bounded below on K. By Proposition 2.1, $\tau(K)<t$, as desired.

(iii)→(iv). Let x be bounded below on $p(H)$ for some $p\in P$ with $\tau(1-p)<t$. Choose $\varepsilon>0$ such that $\|x\xi\|\geq\varepsilon\|\xi\|$ for all $\xi\in p(H)$. Put $L=(1-p)(H)$. With these ε and L, let K be a closed subspace satisfying Lemma A. By Lemma B, we have $\tau(K)\leqq\tau(L)<t$. Since kernel$(x)\subset K$ (See Lemma A), we get the desired conclusion.

(iv)→(iii). Assume that $\nu(x)<t$. Let $E(\cdot)$ be the spectral measure of $|x|$. Since $\lim_{\varepsilon\to 0}\tau(E[0,\varepsilon))=\nu(x)$, there is a positive real number ε such that $\tau(E[0,\varepsilon))<t$. We put $p=E[\varepsilon,\infty)$. Then $\tau(1-p)<t$, while x is bounded below on $p(H)$.

(iii)→(i) Let $p\in P$ be as in (iii). We can find $y\in M$ such that

$$yx\xi=\xi, \text{ for all } \xi\in p(H)$$

and $$yn=0, \text{ for all } \eta\in[x(p(H))]^{\perp}.$$

Then $$yxp=p, \text{ so } yx-I=(yx-I)(I-p)\in I_t,$$

since $I-p\in I_t$.

(v)→(vi) and (vi)→(ii) are clear.

It remains to prove the implication (iv)→(v). As in the proof of (iv)→(iii) we put $p\in E[\varepsilon,\infty)$, where $\varepsilon<0$, $E(\cdot)$ is the spectral projection of $|x|$ and $\tau(1-p)<t$. Let us find $y\in M$ just as in the proof of (iii)→(i) so that $yx-I\in I_t$. We have to show that this y also satisfies that $xy-I\in I_t$.

Let us put $L=x(E[\varepsilon,\infty)(H))$, which is a closed subspace of H such that $L\in M$ and $\tau(L)=\tau(p)=1-t$. For every $\eta\in H$, we write $\eta=\eta_1\oplus\eta_2$, where $\eta_1\in L$ and $\eta_2\in L^{\perp}$. Thus, $\eta_1=x\xi$ for some $\xi\in p(H)$, and $xy\eta=xy\eta_1+xy\eta_2=xyx\xi$ (noticing y in the proof of (iii)→(i) vanishes on $L^{\perp}$, while $yx\xi=\xi$) $=x\xi=\eta_1$. This implies that $xyq=q$, where q is the projection onto L. It follows that $xy-I=(xy-I)(I-q)\in I_t$, since $\tau(q)=\tau(p)$ and hence $\tau(1-q)=\tau(1-p)<t$.

For $x\in M$, $t\in(0,1]$, let us put

$$\sigma_t(x)=\{\lambda\in C : \nu(x-\lambda)\geq t\}.$$

By Proposition 3.1, $\lambda\in\sigma_t(x)$ if and only if $x-\lambda$ is not invertible modulo J_t. In particular, x has no eigenvalue if and only if $x-\lambda$ is invertible modulo J_t for every $t\in(0,1]$ and any $\lambda\in \boldsymbol{C}$.

PROPOSITION 3.2. *The function $x\in M\to\nu(x)\in[0,1]$ is upper semi-continuous with respect to the norm topology of M.*

Proof. To prove the contraposition, let $t\in[0,1]$; $\{x_n\}\subset M$,

$\nu(x_n)\geq t$ and $x_n\rightarrow x$ in norm. It suffices to prove that $\nu(x)\geq t$. Assume contrary that $\nu(x)<t$. By Lemma 3.1, there is $p\in P$ and a positive number ε such that $\tau(1-p)<t$ and $\|xp\xi\|\geq\varepsilon\|p\xi\|$ for all $\xi\in H$. Then, for all $\xi\in H$,

$$\begin{aligned}\|xp\xi\| &= \|xp\xi\|-\|(x_n-x)p\xi\| \\ &\geqq \varepsilon\|p\xi\|-\|x-x\|\|p\xi\| \\ &= (\varepsilon-\|x_n-x\|)\|p\xi\|,\end{aligned}$$

which shows that x_n is bounded below on $p(H)$ for a sufficiently large integer n. By Lemma 3.1 again, we then have $\nu(x_n)<t$, for such n, which is a contradiction as desired.

Lemma 3.3. *Let* $x\in M$, $t\in(0.1]$ *and* $\lambda\in\mathbf{C}$. *If* $\nu_t(x)<|\lambda|$, *then* $x-\lambda$ *is invertible in* M *modulo* J_t.

Proof. Since $\nu_t((1/|\lambda|)x)<1$, we may prove the following: If $\nu_t(x)<1$, then $x-I$ is bounded below on $p(H)$ for some $p\in P$ with $\tau(1-p)<t$ (Proposition 3.1). Since $\nu_t(x)=\inf\{\|xp\| : p\in P : \tau(1-p)<t\}$, there is $p\in P$ such that $\|xp\|<1$ and $\tau(1-p)<t$. Then for all $\xi\in p(H)$ with $\|\xi\|=1$,

$$\begin{aligned}\|(x-I)\xi\| &\geqq \|\xi\|-\|x\xi\|=\|\xi\|-\|xp\xi\| \\ &\geqq \|\xi\|-\|xp\|\|\xi\| \\ &= (1-\|xp\|)\|\xi\|,\end{aligned}$$

while $1-\|xp\|>0$. By Proposition 3.1, $x-I$ is invertible in M modulo J_t, as desired.

Corollary 3.4. *For* $x\in M$, $t\in(0,1]$, $\sigma_t(x)$ *is a compact subset of* $\mathbf{C}$ *contained in the closed disk about the origin with radius* $\nu_t(x)$.

Proof. It is immediate from Proposition 3.2 and Lemma 3.3.

References

1. R.G. Douglas, *On majorization, factorization and range inclusion of operators on Hilbert Space*, Proc. Amer. Math. Soc. **17**(1966) 413-415.
2. G. Edgar, J. Ernest and S.G. Lee, *Weighing operator spectra*, Indiana Univ. Math. J. **21**(1971) 61-80.
3. T. Fack and H. Kosaki, *Generalized s-numbers of τ-measurable operators*,

Pacific J. Math. **123**, No. 2(1986) 269-300.

4. P. Halmos, *A Hilbert space problem book*, Van Nostranel Co. (1967).
5. V. Kaftal, *On the theory of compact operators in von Neumann algebras* I, Indiana Univ. Math. J. **26**(1977) 447-457.
6. S. G. Lee, S. M. Kim and D. P. Chi, *Closed ideals in a semifinite, infinite von Neumann algebra, arising from relative ranks of its elements*, Bull. Korean Math. Soc. **21**(1984) 107-113.
7. S. G. Lee, S. J. Cho and S. K. Kim, *The closed ideals of an infinite semifinite factor*, J. Korean Math. Soc. **22**(1985) 143-149.
8. S. G. Lee and S. J. Cho, *Selfcommutators in an infinite semifinite factor*, J. Korean Math. Soc. **23**(1986) 73-82.

Seoul National University
Seoul 151-742, Korea
and
Song Sim College for Women
Seoul 422-100, Korea