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THE CANONICAL DECOMPOSITION OF SIEGEL MODULAR FORMS I*

MYUNG-HWAN KIM

Introduction

Let $M_k^n(q,\chi)$ be the space of Siegel modular forms of degree n, weight k, level q, and character χ, where n, k, q are positive integers and χ is a Dirichlet character modulo q. The purpose of this article is to show that $M_k^n(q,\chi)$ can be decomposed into $n+1$ subspaces which are pairwise orthogonal with respect to the so called canonical inner product. Actually, we prove this for more general space, namely, $M_k^n(\Gamma,\chi)$, where Γ is any congruence subgroup of the symplectic group $Sp_n(\boldsymbol{Z})$ of level q.

Evdokimov[1], in 1981, gave a proof of this on the way of proving that $M_k^n(\Gamma,\chi)$ has a simultanuous eigenbasis with respect to all the Hecke operators from a certain Hecke ring. But unfortunately, his proof contains a mistake in defining the canonical inner product, and as a consequence his proof of the existence of such eigenbasis needs a major correction.

In this article, the mistakes are corrected to get the canonical decomposition of $M_k^n(\Gamma,\chi)$ (section 3.) and some useful theorems on the decomposition are given.

Let $\boldsymbol{Z}, \boldsymbol{Q}, \boldsymbol{R}$, and $\boldsymbol{C}$ be the ring of rational integers, the field of rational numbers, the field of real numbers, and the field of complex numbers, respectively.

Let $M_{m,n}(\boldsymbol{A})$ be the set of all $m\times n$ matrices over $\boldsymbol{A}$, a commutative ring with 1, and let $M_n(\boldsymbol{A})=M_{n,n}(\boldsymbol{A})$. Let $GL_n(\boldsymbol{A})$ and $SL_n(\boldsymbol{A})$ be the group of invertible matrices in $M_n(\boldsymbol{A})$ and its subgroup consisting of matrices of determinant 1, respectively. For $M\in M_m(\boldsymbol{A})$, $N\in M_{m,n}(\boldsymbol{A})$, let $M[N]={}^tNMN$, where ${}^tN$ is the transpose of N. Let E_n and 0_n

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be the identity and the zero matrices in $M_n(\boldsymbol{A})$, respectively. Let $\det M$ be the determinant of M. For $M \in M_{2n}(\boldsymbol{A})$, we let A_M, B_M, C_M, and D_M be the $n \times n$ block matrices in the upper left, upper right, lower left, and lower right corners of M, respectively, and write $M=(A_M, B_M; C_M, D_M)$. Let N_m be the set of all semi-positive definite (eigenvalues ≥ 0), semi-integral (diagonal entries and twice of nondiagonal entries are integers), symmetric $m \times m$ matrices, and N_m^+ be its subset consisting of positive definite (eigenvalues>0) matrices.

Let $G_n = GSp_n^+(\boldsymbol{R}) = \{M \in M_{2n}(\boldsymbol{R})\ ;\ J_n[M] = rJ_n,\ r>0\}$ where $J_n = (0_n, E_n;\ -E_n, 0_n)$ and $r=r(M)$ is a real number determined by M. Let $\Gamma^n = Sp_n(\boldsymbol{Z}) = \{M \in M_{2n}(\boldsymbol{Z})\ ; J_n[M]=J_n\}$. Let $H_n = \{Z = X+iY \in M_n(\boldsymbol{C})\ ;\ {}^tZ = Z,\ Y>0\}$. For $M \in G_n$ and $Z \in H_n$, we set

$$M(Z) = (A_M Z + B_M)(C_M Z + D_M)^{-1} \in H_n.$$

For $M \in M_n(\boldsymbol{C})$, let $e(M) = \exp(2\pi i \sigma(M))$ where $\sigma(M)$ is the trace of M.

1. Siegel modular forms

Let n, q be positive integers. We define $\Gamma_0^n = \Gamma_0^n(q) = \{M \in \Gamma^n\ ;\ C_M \equiv 0_n \pmod{q}\}$ and $\Gamma_1^n = \Gamma_1^n(q) = \{M \in \Gamma^n : M \equiv E_{2n} \pmod{q}\}$.

Let F be an arbitrary complex valued function on H_n, and let $M = (A, B\ ;\ C, D) \in G_n$. We set

(1.1) $\qquad (F|_k M)(Z) = (\det M)^{k-(n+1)/2} (\det(CZ+D))^{-k} F(M\langle Z \rangle)$

where $Z \in H_n$ and k is a positive integer. Note that $F|_k M$ is holomorphic on H_n if F is. Also note that $F|_k M_1|_k M_2 = F|_k M_1 M_2$ for $M_1, M_2 \in G_n$.

Let $\chi : (\boldsymbol{Z}/q\boldsymbol{Z})^\times \to \boldsymbol{C}^\times$ be a Dirichlet character modulo q. A function $F : H_n \to \boldsymbol{C}$ is called a Siegel modular form of weight k, degree n, level q, and character χ if (i) F is holomorphic on H_n, (ii) $F|_k M = \chi(\det D_M) \cdot F$ for any $M \in \Gamma_0^n$, and (iii) if $n=1$, $(cz+d)^{-k} F((az+b)(cz+d)^{-1})$ is bounded as $\mathrm{Im}(z) \to +\infty$ for any matrix $(a, b\ ;\ c, d) \in \Gamma^1 = SL_2(\boldsymbol{Z})$. The set of all such forms is denoted by $M_k^n(q, \chi)$. This is a finite dimensional vector space over $\boldsymbol{C}$. (See [2].) We define $M_k^n(q)$ by the set of all $F : H_n \to \boldsymbol{C}$ satisfying (i), (iii), and (ii)′ $F|_k M = F$ for any $M \in \Gamma_1^n$. $M_k^n(q)$ is also a finite dimensional vector space over $\boldsymbol{C}$. Note that $M_k^n(q, \chi) \subset M_k^n(q)$.

It is known [3] that every $F \in M_k^n(q)$, hence every $F \in M_k^n(q, \chi)$, has a Fourer expansion of the form

(1. 2) $$F(Z)=\sum_{N\in N_n} f(N)e(NZ/q), \quad Z\in H_n.$$

2. The Siegel operator

Let $0\leq r\leq n$ and let G_r^n be the r-th Satake group ([4], Exposé 12), i. e.,

(2. 1) $$G_r^n=\left\{M\in\Gamma^n ; M=\begin{pmatrix} A_1 & 0 & B_1 & B_{12} \\ A_{21} & A_2 & B_{21} & B_2 \\ C_1 & 0 & D_1 & D_{12} \\ 0 & 0 & 0 & D_2 \end{pmatrix} \text{ with } A_1, B_1, C_1, D_1, \ D_1\in M_r(\boldsymbol{Z})\right\}.$$

The matrix $M_1=(A_1, B_1 ; C_1, D_1)\in\Gamma^r$ and the map $w_r^n : G_r^n\to\Gamma^r$ defined by $w_r^n(M)=M_1$ is a surjective group homomorphism. Note that $w_r^n(\Gamma_i^n\cap G_r^n)=\Gamma_i^n$ for $i=0, 1$.

Let $F : H_n\to \boldsymbol{C}$ be an arbitrary function with Fourier expansion (1. 2). The Siegel operator Φ is defined by

(2. 2) $$(\Phi F)(Z')=\lim_{\lambda\to+\infty} F\begin{pmatrix} Z' & 0 \\ 0 & i\lambda \end{pmatrix}=\sum_{N'\in N_{n-1}} f\begin{pmatrix} N' & 0 \\ 0 & 0 \end{pmatrix} e(N'Z'/q)$$

where $Z'\in H_{n-1}$, $\lambda>0$. For $0\leq r\leq n$, we define Φ^r by Φ^0=the identity operator, $\Phi^r=\Phi\circ\Phi^{r-1}$ for $1\leq r\leq n$. Let $M\in G_r^n$ of the form (2. 1). Then it is easy to see that

(2. 3) $$\Phi^{n-r}(F|_k M)=(\det D_2)^{-k}\ \Phi^{n-r}F|_k(w_r^n(M)).$$

Let $F\in M_k^n(q,\chi)$ and $M_1\in\Gamma_0^r$ and let M be any matrix in $\Gamma_0^n\cap G_r^n$ of the form (2. 1) such that $w_r^n(M)=M_1$. Then from (2. 3) and the condition (ii) follows

(2. 4) $$(\Phi^{n-r}F)|_k M_1=(\det D_2)^k\ \chi(\det D_M)\Phi^{n-r}F.$$

So $\Phi^{n-r}F\in M_k^r(q,\chi_r)$ where $\chi_r(\det D_1)=(\det D_2)^k\ \chi(\det D_M)$ which is independent of the choice of M according to (2. 3). Moreover, since we can choose M such that $\det D_M=\det D_1$ and $\det D_2=1$, we get

(2. 5) $$\Phi^{n-r}F\in M_k^r(q,\chi) \text{ if } F\in M_k^n(q,\chi).$$

Similar argument shows that

(2. 6) $$\Phi^{n-r}\in M_k^r(q) \text{ if } F\in M_k^n(q).$$

Let Γ be a congruence subgroup of level q, i. e., $\Gamma_1^n\subset\Gamma\subset\Gamma^n$.
Let χ be a character: $\Gamma\to\boldsymbol{C}^\times$ such that $\chi(\Gamma_1^n)=1$. We set $M_k^n(\Gamma,\chi)$ to be the set of all $F : H_n\to\boldsymbol{C}$ satisfying the conditions (i), (iii),

and (ii)″ $F|_kM=\chi(M)F$ for any $M\in\Gamma$. This is also a finite dimensional vector space over $\boldsymbol{C}$. So under this definition, Siegel modular spaces $M_k^n(q,\chi)$ and $M_k^n(q)$ can be identified with $M_k^n(\Gamma_0^n,\chi)$ and $M_k^n(\Gamma_1^n,\chi_0)$ where χ_0 is the trivial character. For convenience we set $M_k^0(\Gamma,\chi)=\boldsymbol{C}, \Gamma^0=1$, and H_0 to be a single point.

Let $F\in M_k^n(\Gamma,\chi)$ and $M_1\in w_r^n(\Gamma\cap G_r^n)$. Take any $M\in\Gamma\cap G_r^n$ of the form (2.1) such that $w_r^n(M)=M_1$. Then

(2.7) $$\Phi^{n-r}F\in M_k^r(\Gamma_r,\chi_r)$$

where $\Gamma_r=w_r^n(\Gamma\cap G_r^n)$ and χ_r is a character on $\Gamma\cap G_r^n$ defined by $\chi_r(M_1)=(\det D_2)^k\,\chi(M)$. Again (2.3) guarantees the independence of χ_r under the choice of M.

Let $M\in\Gamma^n$. Then

(2.8) $$F|_kM\in M_k^n(\Gamma^M,\chi^M)$$

where $\Gamma^M=M^{-1}\Gamma M$ and χ^M is a character on $M^{-1}\Gamma M$ defined by $\chi^M(\hat{M})=\chi(M\hat{M}M^{-1})$ for $\hat{M}\in M^{-1}\Gamma M$. Combining (2.7) and (2.8), we get

(2.9) $$\Phi^{n-r}(F|_kM)\in M_k^r(\Gamma_r^M,\chi_r^M)$$

where $\Gamma_r^M=w_r^n(M^{-1}\Gamma M\cap G_r^n$ and $\chi_r^M=(\chi^M)_r$.

We denote $M_k^n((\Gamma_0^n)^M,\chi^M)$ and $M_k^r((\Gamma_0^n)_r^M,\chi_r^M)$ by $M_k^n(q^M,\chi^M)$ and $M_k^r(q_r^M,\chi_r^M)$, respectively. (2.6) shows that $M_k^r((\Gamma_0^n)_r,\chi_r)=M_k^r(q,\chi)$. It's easy to see that $M_k^n((\Gamma_1^n)^M,\ \chi^M)=M_k^n(q)$ and $M_k^r((\Gamma_1^n)_r^M,\ (\chi_0)_r^M)=M_k^r((\Gamma_1^n)_r,\ (\chi_0)_r)=M_k^r(q)$.

3. The canonical decomposition

$F\in M_k^n(\Gamma,\chi)$ is called a cusp form if $\Phi(F|_kM)=0$ for all $M\in\Gamma^n$. For $F,G\in M_k^n(\Gamma,\chi)$, we set

(3.1) $$(F,G)_o=\int_{D(\Gamma)}F(Z)\overline{G(Z)}\,(\det Y)^k d\tilde{Z}$$

where $D(\Gamma)$ is a fundamental domain of Γ in H_n, $Z=X+iY\in H_n$, and $d\tilde{Z}=(\det Y)^{-n-1}dXdY$ is the G_n-invariant volume element on H_n. If either F or G is a cusp form, then the pairing (3.1) is a well defined non-degenerate Hermitian inner product ([4], Exposé 7) and is called the Maass-Petersson inner product on $M_k^n(\Gamma,\chi)$. But otherwise, the pairing (3.1) is meaningless.

We now construct a positive definite Hermitian inner product which is meaningful on the whole space $M_k^n(\Gamma, \chi)$.

Let $G_k^n(\Gamma, \chi)$ be the subspace of $M_k^n(\Gamma, \chi)$ consisting of all the cusp forms. If $F \in M_k^n(\Gamma, \chi)$, then F can be written uniquely in the form $F = F' + F_n$ where $F_n \in G_k^n(\Gamma, \chi)$ and F' is contained in the orthogonal complement of $G_k^n(\Gamma, \chi)$ in $M_k^n(\Gamma, \chi)$ with respect to the Maass–Petersson inner proudct. We call F_n the cusp part of F. We set

$$(3.2) \qquad (F, G) = \sum_{r=0}^{n} \sum_{M \in \Gamma \backslash \Gamma^n} [\Gamma^r : \Gamma_r^M]^{-1} \Big((\Phi_M^{n-r} F)_r, (\Phi_M^{n-r} G)_r \Big)_o$$

where $\Phi_M^{n-r} F = \Phi^{n-r}(F|_k M)$, $(\Phi_M^{n-r} F)_r$ is the cusp part of $\Phi_M^{n-r} F$, and $(-, -)_o$ is the Maass–Petersson inner product on the space $M_k^r(\Gamma_r^M, \chi_r^M)$.

THEORME 3.1. *The pairing* (3.2) *is a well defined positive definite Hermitian inner product on the whole space* $M_k^n(\Gamma, \chi)$, *which is called the canonical inner product on the space.*

Proof. Since Γ^M, χ^M are independent of the choice of representative M of the left coset ΓM with $M \in \Gamma^n$, so are the index $[\Gamma^r : \Gamma_r^M]$ and the space $M_k^r(\Gamma_r^M, \chi_r^M)$. The Maass–Petersson inner product $(-, -)_o$ is also independent of the choice of M because $|\chi(M')| = 1$ for any $M' \in \Gamma$. So (3.2) is a well defined Hermitian inner product. The positive definiteness follows immediately from (2.2) and the obvious fact that $(-, -)_o$ is positive definite when restricted to cusp forms. The theorem is proved.

We now decompose $M_k^n(\Gamma, \chi)$ into $n+1$ mutually orthogonal subspaces with respect to the canonical inner product. For $0 \le r \le n$, we set

$$(3.3) \quad M_k^{n,r}(\Gamma, \chi) = \Big\{ F \in M_k^n(\Gamma, \chi) \ ; \ \Phi_M^{n-r} F \text{ is a cusp form for every } M \in \Gamma^n \text{ such that } \Big(F, \sum_{s=r+1}^{n} M_k^{n,s}(\Gamma, \chi)\Big) = 0 \ \textit{if} \ r \ne n \Big\}.$$

Observe that $M_k^{n,n}(\Gamma, \chi) = G_k^n(\Gamma, \chi)$.

THEOREM 3.2. *For* $0 \le r \le n-1$,

$$(3.4) \quad M_k^{n,r}(\Gamma, \chi) = \Big\{ F \in M_k^n(\Gamma, \chi) \ ; \ \Phi_M F \in \sum_{s=r}^{n-1} M_k^{n-1,s}(\Gamma_{n-1}^M, \chi_{n-1}^M) \ \textit{such that} \ \Big(F, \sum_{s=r+1}^{n} M_k^{n,s}(\Gamma, \chi)\Big) = 0 \Big\}.$$

Proof. Let the right sides of (3.3) and (3.4) be A and B. We use induction on n. For $n=1$ it is well known [5] that the lemma holds. Let $n>1$. It is clear that $B\subset A$. Suppose $F\in A-B$. Then there exists M such that $\Phi_M F$ is not contained in $\sum_{s=r}^{n-1,s} M_k^{n-1}(\Gamma_{n-1}^M, \chi_{n-1}^M)$. But then from the induction hypothesis and (2.3) follows that $\Phi_{M'}^{n-r}F$ is not a cusp form for some $M'\in\Gamma^n$, which is impossible, whence $A\subset B$.

When $n=1$, $M_k^{1,0}(\Gamma,\chi)$ and $M_k^{1,1}(\Gamma,\chi)$ coincide with the subspaces of classical Eisenstein series [5] and cusp forms [6]. Furthermore, they are orthogonal to each other with respect to Maass–Petersson inner product as well as to the canonical inner product. This can be generalized for arbitrary n. More precisely.

THEOREM 3.3. *The space $M_k^n(\Gamma,\chi)$ is decomposed into $n+1$ subspaces $M_k^{n,r}(\Gamma,\chi)$, $0\leq r\leq n$, which are pairwise orthogonal with respect to the canonical inner product.*

Proof. From Theorem 3.2. and induction on n, the theorem follows.

Maass[7] proved this for $M_k^n(\Gamma,\chi_0)$.

We write $M_k^n(\Gamma,\chi)=\perp_{r=0}^{n} M_k^{n,r}(\Gamma,\chi)$ and call it the canonical decomposition of $M_k^n(\Gamma,\chi)$. The subspace $M_k^{n,r}(\Gamma,\chi)$ is called the r-th canonical subspace of $M_k^n(\Gamma,\chi)$ for each $r=0,\dots,n$.

4. Some theorems

Let $F, G\in M_k^n(\Gamma,\chi)$ such that at least one of which is a cusp form, say, G. Since $\Phi_M^s G=0$ for $s>0$,

$$(4.1)\qquad (F,G)=\sum_{M\in\Gamma\backslash\Gamma^n}[\Gamma^n:\Gamma^M]^{-1}\Big((F|_kM)_n, (G|_kM)_n\Big)_0.$$

THEOREM 4.1. *If $F\in M_k^n(\Gamma,\chi)$, then $F'|_kM=(F|_kM)'$ and $F_n|_kM=(F|_kM)_n$ for any $M\in\Gamma^n$.*

Proof. The second equality follows from the definition of a cusp form. For $F, G \in M_k^n(\Gamma, \chi)$, at least one of which is a cusp form, it is easy to see that $(F|_k M, G|_k M)_o = (F, G)_o$ where the former pairing is the Mass–Petersson inner product on $M_k^n(\Gamma^M, \chi^M)$ and the latter is that on $M_k^n(\Gamma, \chi)$. The first equality follows.

THEOREM 4.2. *For $F, G \in M_k^n(\Gamma, \chi)$, at least one of which is a cusp form, the canonical inner product coincide with the Maass–Petersson inner product.*

Proof. Let G be a cusp form. Since $((F|_k M)_n, (G|_k M)_n)_o = (F_n|_k M, G_n|_k M)_o = (F_n, G_n)_o = (F_n, G)_o = (F_n, G)_o + (F', G)_o = (F, G)_o$, from (4.1) we have $(F, G) = \sum_{M \in \Gamma\backslash\Gamma^n} [\Gamma^n : \Gamma^M]^{-1}(F, G)_o$. $[\Gamma^n : \Gamma^M] = [\Gamma^n : \Gamma]$ for any $M \in \Gamma^n$. The theorem is proved.

Let $M \in \Gamma^n$ be given. Let $T_M : M_k^n(\Gamma, \chi) \to M_k^n(\Gamma^M, \chi^M)$ be a homomorphism defined by $T_M(F) = F|_k M$. It is easy to see that T_M is an isomorphism that preserves the canonical inner product and hence the canonical decomposition, i. e.,

$$(4.2)\qquad (F, G) = (F|_k M, G|_k M) = (T_M F, T_M G)$$

where the left canonical inner product is on $M_k^n(\Gamma, \chi)$ and the right is on $M_k^n(\Gamma^M, \chi^M)$, and

$$(4.3)\qquad T_M(M_k^n(\Gamma, \chi)) = M_k^n(\Gamma^M, \chi^M).$$

Let Γ' be a congruence subgroup contained in Γ and let χ' be the restriction of χ to Γ'. Then $M_k^n(\Gamma, \chi) \subset M_k^n(\Gamma', \chi')$.

THEOREM 4.3. *For $F, G \in M_k^n(\Gamma, \chi)$,*

$$(4.4)\qquad (F, G) = [\Gamma : \Gamma']^{-1}(F, G)'$$

where the left canonical inner product is on $M_k^n(\Gamma, \chi)$ and the right is on $M_k^n(\Gamma', \chi')$.

Proof. From (3.1) follows that $(F, G)_o = [\Gamma : \Gamma']^{-1}(F, G)_o'$ where $(-, -)_o'$ is the Maass–Petersson inner product on $M_k^n(\Gamma', \chi')$. Let $\{N_i\}_{i=1,2,\ldots,m} \subset \Gamma$, $\{M_j\}_{j=1,2,\ldots,l} \subset \Gamma^n$ be full sets of left coset representatives of $\Gamma'\backslash\Gamma$, $\Gamma\backslash\Gamma^n$. Then $\{N_i M_j\}$ is a full set of left coset representatives of $\Gamma'\backslash\Gamma^n$. For each $0 \leq r \leq n$, $N \in \{N_i\}$, $M \in \{M_j\}$, we have $(\Phi_{NM}^{n-r} F)_r = (\Phi^{n-r} F|_k NM)_r = \chi(N)(\Phi_M^{n-r} F)_r$. Similary, $(\Phi_{NM}^{n-r} G)_r$

$=\chi(N)(\Phi_M^{n-r}G)_r$. Since $(-,-)_o'$ is Hermitian and $|\chi(N)|=1$, $((\Phi_{NM}^{n-r}F)_r, (\Phi_{NM}^{n-r}G)_r)_o'=((\Phi_M^{n-r}F)_r, (\Phi_M^{n-r}G)_r)_o'$. So $[\Gamma^r : (\Gamma')_r^{NM}]^{-1}((\Phi_{NM}^{n-r}F)_r, (\Phi_{NM}^{n-r}G)_r)_o'=[\Gamma^r : \Gamma_r^M]^{-1}((\Phi_M^{n-r}F)_r, (\Phi_M^{n-r}G)_r)_o$ and hence from (3. 2) follows $(F,G)'=m(F,G)=[\Gamma:\Gamma'](F,G)$ which proves the theorem.

THEOREM 4. 4. *For each* $0\leq r\leq n$, *we have*

(4. 5) $$M_k^{n,r}(\Gamma,\chi)=M_k^{n,r}(\Gamma',\chi')\cap M_k^n(\Gamma,\chi),$$

Proof. We use induction on $s=n-r$. For $s=0$ $(n=r)$, (4. 5) follows immediately from (3. 3) and Theorem 4. 3. Let F be in the right side of (4. 5) and $G\in\sum_{s=r+1}^{n}M_k^{n,s}(\Gamma,\chi)$ for $r<n$. From induction hypothesis $(F,G)=[\Gamma:\Gamma']^{-1}(F,G)'=0$. So from (3. 3), $F\in M_k^{n,r}(\Gamma,\chi)$. To show the reverse inclusion, let $F\in M_k^{n,r}(\Gamma,\chi)$. Then $F\in\sum_{s=r}^{n}M_k^{n,s}(\Gamma',\chi')$. Write $F=F_r+G$ where $F_r\in M_k^{n,r}(\Gamma',\chi')$ and $G\in\sum_{s=r+1}^{n}M_k^{n,s}(\Gamma',\chi')$. From induction hypothesis $0=(F,G)'=(F_r+G,G)'=(G,G)'$. So $G=0$ and $F=F_r\in M_k^{n,r}(\Gamma',\chi')$. The theorem follows.

According to equalities (4. 2), (4. 3) and Theorems 4. 3., 4. 4., when one needs to prove a certain property related to the canonical inner product and decomposition on $M_k^n(\Gamma,\chi)$, in particular, on $M_k^n(q,\chi)$, it suffices to prove it for $M_k^n(q)$.

Finally, we prove the invariance of the r-th canonical subspace of $M_k^n(\Gamma,\chi)$ under the action $|_kM$ for $M\in\Gamma^n$.

THEOREM 4. 5. *If F is in $M_k^{n,r}(\Gamma,\chi)$, then so is $F|_kM$ for any $M\in\Gamma^n$.*

Proof. As the remark above, it is enough to show the theorem for $F\in M_k^n(q)$. It is clear that $F|_kM\in M_k^n(q)$. Again we use induction on $s=n-r$. $\Phi_{M'}^{n-r}F$ is a cusp form for each $M'\in\Gamma^n$. If M' runs over Γ^n, then so does MM'. So $\Phi_{MM'}^{n-r}F=\Phi_{M'}^{n-r}(F|_kM)$ is also a cusp form for each $M\in\Gamma^n$ and hence it suffices to show

(4. 6) $$(F|_kM,G)=0 \text{ for any } G\in\sum_{s=r+1}^{n}M_k^{n,s}(q) \text{ for } 0\leq r<n.$$

From (3. 2) we get

(4. 7) $$(F|_kM,G)=\sum_{r=0}^{n}\sum_{M'\in\Gamma_1^n\backslash\Gamma^n}[\Gamma^r:\Gamma_1^r]^{-1}\Big((\Phi_{M'}^{n-r}(F|_kM))_r,\ (\Phi_{M'}^{n-r}G)_r\Big)_o,$$

where $(-,-)_o$ is the Maass–Petersson inner product on $M_k^r(q)$, because $(\Gamma_1^n)_r^{M'}=\Gamma_1^r$. If M' runs over a full set of representatives of $\Gamma_1^n\backslash\Gamma^n$, then so dose $M^{-1}M'$. Substitution of M' by $M^{-1}M'$ in (4.7) yields $(F|_kM,G)=(F,G|_kM^{-1})$. From induction hypothesis $(F,G|_kM^{-1})=0$. So (4.6) and hence the theorem follows.

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Seoul National University
Seoul 151-742, Korea