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THE FINITE HANKEL-SCHWARTZ TRANSFORM

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1. Introduction

A. L. Schwartz [10] introduced the following variant of the Hankel transform

$$F(y)=\int_0^\infty x^{2\nu+1}\mathcal{J}_\nu(xy)f(x)\,dx \quad , \tag{1.1}$$

where $\mathcal{J}_\nu(x)=x^{-\nu}J_\nu(x)$, $J_\nu(x)$ being the Bessel function of the first kind of order ν. This transform has been investigated in spaces of generalized functions by different authors [1, 3, 9] and was called the Hankel-Schwartz transformation by W. Y. Lee [6].

In this paper we study its finite version, first from a point of view entirely classical. For it we begin by considering a Fourier-Bessel type of series expansion which suggests the definition of the classical finite Hankel-Schwartz transformation. Later this transform is extended to spaces of generalized functions. In order to do that, we modify previously in a natural way the method developed by Zemanian [15] in his research on a variety of distributio nal series expansions. Recall that the success of Zemanian's method lies in the fact that the differential operators considered are always selfadjoint. The main objective of our work is to give a procedure that turns out to be valid for more general operators. For this purpose, two differential operators having the same positive eigenvalues and whose respective systems of eigenfunctions verify the same orthogonality condition, are simultaneously considered. Then two testing function spaces and their duals are constructed such that certain Fourier-Bessel series converge in them. So we are led on to define two new generalized transforms, which will be called the finite Hankel-Schwartz integral transformations of the first kind of order ν. This approach is reminiscent of the procedure described in [8] for extending the infinite Hankel transformation to distributional

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spaces.

Finally, the operational calculus generated is used in solving certain partial differential equations.

2. Preliminary Results

Consider the Sturm–Liouville problem [11, p. 440]

$$(\Delta_\nu+\lambda^2)y=0 \tag{2.1}$$

$$a_1 y(a)+a_2 y'(a)=0,\ b_1 y(b)+b_2 y'(b)=0 \tag{2.2}$$

where $\Delta_\nu=x^{-2\nu-1}Dx^{2\nu+1}D$; a, b, a_1, a_2, b_1, b_2 are real constants and $D=d/dx$.

The general solution of (2.1) is

$$y=\phi_\lambda(x)=A(\lambda)\mathcal{J}_\nu(\lambda x)+B(\lambda)\mathcal{Y}_\nu(\lambda x), \tag{2.3}$$

where $\mathcal{J}_\nu(x)=x^{-\nu}J_\nu(x)$ and $\mathcal{Y}_\nu(x)=x^{-\nu}Y_\nu(x)$, $Y_\nu(x)$ denoting the Bessel function of the second kind of order ν.

Equation (2.1) may be written as

$$x^{-2\nu}\frac{d}{dx}\left(x^{2\nu+1}\frac{dy}{dx}\right)^2+\lambda^2x^{2\nu+2}\frac{d}{dx}(y^2)=0$$

Upon integrating by parts in the interval [a, b] we obtain

$$2\lambda^2\int_a^b x^{2\nu+1}y^2dx-[x^{2\nu+1}(xy'^2+\lambda^2xy^2+2\nu yy')]_a^b=0 \tag{2.4}$$

Let $y=\phi_n(x)$ be the eigenfunctions of the problem (2.1)~(2.2) which correspond to the non-zero eigenvalues λ_n. Thus the orthogonality condition

$$\int_a^b x^{2\nu+1}\phi_m(x)\phi_n(x)\,dx=\begin{cases}\frac{1}{2\lambda_n^2}[x^{2\nu+1}\{x(\phi_n'(x))^2+\lambda_n^2x\phi_n^2(x)\\ \qquad +2\nu\phi_n(x)\phi_n'(x)\}]_a^b, & m=n\\ 0, & m\neq n\end{cases} \tag{2.5}$$

may be derived from (2.4) and Sturm–Liouville general theory.

Consider now the problem

$$(\Delta_\nu+\lambda^2)\phi(x)=0,\quad 0\leqq x\leqq a, \tag{2.6}$$
$$\phi(a)=0$$

whose solution is in view of (2.3)

$$\phi_n(x)=\mathcal{J}_\nu(j_nx), \tag{2.7}$$

where $j_1, j_2, \ldots$ represent the positive zeros arranged in ascending order of magnitude of the equation [13, p. 479]

$$\mathcal{J}_\nu(j_na)=0 \tag{2.8}$$

The above orthogonality condition (2.5) now becomes

$$\int_0^a x^{2\nu+1}\mathcal{J}_\nu(j_m x)\mathcal{J}_\nu(j_n x)\,dx=\begin{cases}\dfrac{j_n^2 a^{2\nu+4}}{2}\mathcal{J}_{\nu+1}^2(j_n a), & m=n\\ 0 & , m\neq n\end{cases} \tag{2.9}$$

3. The Fourier-Bessel series and the classical finite Hankel-Schwartz transformation

Let $f(x)$ be an arbitrary function defined in the interval $(0, a)$. This function may be formally expressed by virtue of (2.9) as the following Fourier-Bessel expansion

$$f(x)=\sum_{m=1}^{\infty} a_m \mathcal{J}_\nu(j_m x) \tag{3.1}$$

where

$$a_m=\frac{2}{j_m^2 a^{2\nu+4}\mathcal{J}_{\nu+1}^2(j_m a)}\int_0^a x^{2\nu+1}\mathcal{J}_\nu(j_m x)f(x)\,dx \tag{3.2}$$

From now on it will be assumed that $a=1$ for the sake of the simplicity.

To study the convergence of the series (3.1), consider the partial sum

$$S_n(x)=\sum_{m=1}^{n} a_m \mathcal{J}_\nu(j_m x) \tag{3.3}$$

If $P_n(x, t)$ denotes the finite sum

$$P_n(x, t)=\sum_{m=1}^{n}\frac{2\mathcal{J}_\nu(j_m x)\mathcal{J}_\nu(j_n t)}{j_m^2\mathcal{J}_{\nu+1}^2(j_m)} \tag{3.4}$$

(3.3) may be written as

$$S_n(x)=\int_0^1 t^{2\nu+1}P_n(x, t)f(t)\,dt$$

We now establish that

$$\lim_{n\to\infty}\int_0^1 t^{2\nu+1}P_n(x, t)\,dt=1,\quad 0<x<1 \tag{3.5}$$

First of all note that from (3.4) we have

$$\int_0^1 t^{2\nu+1}P_n(x, t)\,dt=\sum_{m=1}^{n}\frac{2\mathcal{J}_\nu(j_m x)}{j_m^2\mathcal{J}_{\nu+1}(j_m)}$$

Then, if C denotes the rectangle with vertices at $\pm Bi$, $A_n\pm Bi$, where $j_n<A_n<j_{n+1}$, an application of residues theory yields.

$$\oint_C \frac{2\mathcal{J}_\nu(wx)}{w\mathcal{J}_\nu(w)}dw=-2\pi i\sum_{m=1}^{n}\frac{2\mathcal{J}_\nu(j_m x)}{j_m^2\mathcal{J}_{\nu+1}(j_m)}.$$

Since the integrand is an odd function of w, the integral along the side on the imaginary axis reduces to the contribution in a small indentation at the origin, whose value is $2\pi i$. Thus, when $B\to\infty$, we get

$$\sum_{n=1}^{m}\frac{2\mathcal{J}_\nu(j_m x)}{j_m^2\mathcal{J}_{\nu+1}(j_m)}=1-\frac{1}{2\pi i}\int_{A_n-\infty i}^{A_n+\infty i}\frac{2\mathcal{J}_\nu(xw)}{w\mathcal{J}_\nu(w)}dw.$$

Now (3.5) follows immediately as $n\to\infty$. In a similar way it can be stated that

$$\lim_{n\to\infty}\int_0^x t^{2\nu+1}P_n(x,t)\,dt=\frac{1}{2},\quad 0<x<1, \tag{3.6}$$

$$\lim_{n\to\infty}\int_x^1 t^{2\nu+1}P_n(x,t)\,dt=\frac{1}{2},\quad 0<x<1. \tag{3.7}$$

Proceeding as in [13, p. 584] we obtain the bounds

$$|P_n(x,t)|\leqq\frac{K(xt)^{-\nu-\frac{1}{2}}}{|t^2-x^2|\,(2-x-t)} \tag{3.8}$$

and

$$\left|\int_0^t t^{2\nu+1}P_n(x,t)\,(t^2-x^2)\,dt\right|\leqq\frac{K}{A_n(2-x-t)}\left(\frac{t}{x}\right)^{\nu+\frac{1}{2}} \tag{3.9}$$

where K is a positive constant.

Taking into account (3.8) and (3.9), a result analogous to the Riemann-Lebesgue Lemma can be established:

Let $\nu\geqq-\frac{1}{2}$ *and* $0\leqq c<d\leqq1$. *If* $f(t)$ *is absolutely integrable in* (c, d) *and* $x\notin(c, d)$, *then*

$$\lim_{n\to\infty}\int_c^d t^{2\nu+1}P_n(x,t)f(t)\,dt=0,\quad 0<x<1 \tag{3.10}$$

We next analyse the identity

$$\int_0^1 t^{2\nu+1}P_n(x,t)f(t)\,dt-f(x-0)\int_0^x t^{2\nu+1}P_n(x,t)\,dt-f(x+0).$$

$$\int_x^1 t^{2\nu+1}P_n(x,t)\,dt=\int_0^x t^{2\nu+1}[f(t)-f(x-0)]P_n(x,t)\,dt$$

$$+\int_x^1 t^{2\nu+1}[f(t)-f(x+0)]P_n(x,t)\,dt \tag{3.11}$$

As $n\to\infty$ the left-hand side of (3.11) tends to

$$\lim_{n\to\infty}S_n(x)-\frac{1}{2}[f(x+0)+f(x-0)]$$

because of (3.3), (3.6) and (3.7). On the other hand, (3.10) and the same argument developed in [13, p. 592] may be invoked to show that the right-hand side vanishes as $n\to\infty$. This result can be summarized as follows

THEOREM 1. *Let $f(x)$ be a function defined and absolutely integrable on* $(0, 1)$. *Assume that* $\nu \geqq -\frac{1}{2}$ *and set*

$$a_m = \frac{2}{j_m^2 \mathcal{J}_{\nu+1}^2(j_m)} \int_0^1 t^{2\nu+1} \mathcal{J}_\nu(j_m t) f(t)\, dt, \quad m=1, 2, \ldots$$

If $f(t)$ *is of bounded variation in* (c, d) $(0<c<d<1)$ *and if* $x \in (c, d)$, *then the series*

$$\sum_{m=1}^{\infty} a_m \mathcal{J}_\nu(j_m x)$$

converges to $\frac{1}{2}[f(x+0)+f(x-0)]$.

Expressions (3.1) and (3.2) and Theorem 1 suggest to introduce the integral transform

$$(h_{1,\nu} f)(n) = (h_\nu f)(n) = F_\nu(n) = \int_0^a x^{2\nu+1} \mathcal{J}_\nu(j_n x) f(x)\, dx \qquad (3.12)$$

which will be called the finite Hankel-Schwartz integral transformation of the first kind. Its inversion formula is given by

$$(h_\nu^{-1} F_\nu)(x) = f(x) = \frac{2}{a^{2\nu+4}} \sum_{n=1}^{\infty} \frac{F_\nu(n) \mathcal{J}_\nu(j_n x)}{j_n^2 \mathcal{J}_{\nu+1}^2(j_n a)} \qquad (3.13)$$

We point out the following operational rules

(i) If $f(x) \in C^2(0, a)$, upon integrating by parts we deduce the relation

$$h_\nu\left[f''(x) + \frac{2\nu+1}{x} f'(x)\right] = j_n^2 a^{2\nu+2} \mathcal{J}_{\nu+1}(j_n a) f(a) - j_n^2 h_\nu[f(x)] \qquad (3.14)$$

(ii) If $f(x) \in C^{2r}(0. a), f^{(i)}(0)$ are finite and $f^{(i)}(a) = 0$ $(i=0, 1, 2, \ldots, 2r-2)$, then

$$h_\nu\left[\left\{f''(x) + \frac{2\nu+1}{x} f'(x)\right\}^r\right] = (-j_n^2)^r h_\nu[f(x)],$$

r being a positive integer.

4. The testing function spaces A_ν and A_ν^* and their duals

In this section we shall employ the same notation and terminology as those used in [15]. Thus, I denotes the interval $(0, 1)$ and ν will be always restricted to the interval $-\frac{1}{2} \leqq \nu < \infty$. $L_2(I)$ and $L_2^*(I)$

represent the spaces of equivalence classes of functions that are quadratically integrable on I with regard to the weight functions $x^{2\nu+1}$ and $x^{-2\nu-1}$, respectively. A mixed inner product is defined on $L_2(I) \times L_2^*(I)$ by

$$(f, g)=\int_I f(x)\overline{g(x)}\ dx,\quad f\in L_2(I), g\in L_2^*(I) \tag{4.1}$$

where $\overline{g(x)}$ denotes the complex conjugate of $g(x)$. This definition is consistent with the inner product considered usually on $L_2(I)$ and $L_2^*(I)$. Indeed, (4.1) can be rewritt

$$\begin{aligned}(f, g) &=\int_I x^{2\nu+1}f(x)\,(x^{-2\nu-1}\overline{g(x)})\,dx\\ &=\int_I x^{-2\nu-1}(x^{2\nu+1}f(x))\overline{g(x)}\,dx,\end{aligned}$$

and note that $f(x)$ and $x^{-2\nu-1}g(x)$ belong to $L_2(I)$, while $x^{2\nu+1}f(x)$ and $g(x)$ are in $L_2^*(I)$.

$D(I)$, $E(I)$, $D'(I)$ and $E'(I)$ are well-known spaces of testing functions and their duals [16, p. 32].

The differential operator

$$\Delta_{\nu,x}=\Delta_\nu=D^2+\frac{2\nu+1}{x}D=x^{-2\nu-1}Dx^{2\nu+1}D \tag{4.2}$$

is not selfadjoint. We consider, together with Δ_ν, the operator

$$\Delta_{\nu,x}^*=\Delta_\nu^*=D^2-\frac{2\nu+1}{x}D+\frac{2\nu+1}{x^2}=Dx^{2\nu+1}Dx^{-2\nu-1}. \tag{4.3}$$

Δ_ν^* is called the adjoint operator of Δ_ν. Note that

$$\Delta_\nu^*=x^{2\nu+1}\Delta_\nu x^{-2\nu-1} \tag{4.4}$$

The functions

$$\phi_n(x)=\mathcal{J}_\nu(j_n, x),\quad n=1, 2, \ldots, \tag{4.5}$$

are the eigenfunctions of Δ_ν, whereas the functions

$$\phi_n^*(x)=x^{2\nu+1}\mathcal{J}_\nu(j_n x),\quad n=1, 2, \ldots \tag{4.6}$$

are the eigenfunctions of Δ_ν^*. In both cases we have the same eigenvalues j_n $(n=1, 2, \ldots)$, which are the positive roots of equation (2.8) with $a=1$. Therefore,

$$(\Delta_\nu+j_n^2)\phi_n(x)=0$$

and

$$(\Delta_\nu^*+j_n^2)\phi_n^*(x)=0 \tag{4.7}$$

Systems of eigenfunctions $\{\phi_n(x)\}_{n=1}^\infty$ and $\{\phi_n^*(x)\}_{n=1}^\infty$ verify by (2.9) and (4.1) the orthogonality condition

$$(\phi_n, \phi_m^*) = (\phi_m^*, \phi_n) = \begin{cases} 0 & \text{, if } m \neq n \\ \frac{j_n^2}{2} \mathcal{J}_{\nu+1}^2(j_n) & \text{, if } m = n \end{cases} \tag{4.8}$$

This is equivalent to say that $\{\phi_n(x)\}_{n=1}^{\infty}$ is orthogonal with respect to the weight function $x^{2\nu+1}$ and, on the other hand, that $\{\phi_n^*(x)\}_{n=1}^{\infty}$ is orthogonal with respect to $x^{-2\nu-1}$. In any case (4.8) holds.

A_ν is defined as the linear space of all infinitely differentiable complex-valued functions $\phi(x)$ on I such that

(i) $$\alpha_{k,\nu}(\phi) = \left[\int_I x^{-2\nu-1} |\Lambda_\nu^{*k} \phi(x)|^2 dx\right]^{\frac{1}{2}}$$

exists for every $k=0, 1, 2, \ldots$

(ii) $$(\Lambda_\nu^{*k}\phi, \phi_n) = (\phi, \Lambda_\nu^k \phi_n)$$

holds for each n and k.

A_ν is the countable multinormed space having the topology generated by $\{\alpha_{k,\nu}\}$. A_ν is also complete. Consequently, A_ν is a Fréchet space.

In our context we can establish a result analogous to [15, Lemma 1]

THEOREM 2. *Let* $\nu \geqq -\frac{1}{2}$. *Every member* $\phi \in A_\nu$ *can be expanded into a series of the form*

$$\phi = \sum_{n=1}^{\infty} \frac{2}{j_n^2 \mathcal{J}_{\nu+1}^2(j_n)} (\phi, \phi_n) \ \phi_n^*, \tag{4.9}$$

which converges in A_ν.

Proof. Note that $\Lambda_\nu^{*k}\phi \in L_2^*(I)$. Hence, by (ii) and (4.7) we have

$$\Lambda_\nu^{*k}\phi = \sum_{n=1}^{\infty} \frac{2}{j_n^2 \mathcal{J}_{\nu+1}^2(j_n)} (\Lambda_\nu^{*k}\phi, \phi_n)\phi_n^* =$$

$$= \sum \frac{2}{j_n^2 \mathcal{J}_{\nu+1}^2(j_n)} (\phi, \Lambda_\nu^k \phi_n)\phi_n^* = \sum \frac{2}{j_n^2 \mathcal{J}_{\nu+1}^2(j_n)} (\phi, \phi_n) \Lambda_\nu^{*k}\phi_n^*$$

where the series involved converge in $L_2^*(I)$. Therefore,

$$\alpha_{k,\nu}\left[\phi - \sum_{n=1}^{m} \frac{2}{j_n^2 \mathcal{J}_{\nu+1}^2(j_n)} (\phi, \phi_n)\phi_n^*\right] \to 0$$

as $m \to \infty$. This completes the proof of Theorem 2.

A_ν' is the dual space of A_ν and is too complete. We now list some properties of these spaces

(a) $D(I) \subset A_\nu \subset E(I)$. $E'(I)$ is a subspace of A_ν'.

(b) It can be seen that every eigenfunction ϕ_n^*, given by (4.6), belongs to A_ν.

(c) The operation $\phi \to \Lambda_\nu^* \phi$ is a continuous linear mapping of A_ν into itself. Consequently, the operation $f \to \Lambda_\nu f$ defined on A_ν' by

$$(\Lambda_\nu f, \phi) = (f, \Lambda_\nu^* \phi) \tag{4.10}$$

is also a continuous linear mapping of A_ν' into itself.

We will have need of another testing function space A_ν^* along this work. A_ν^* consists of all infinitely differentiable functions $\phi(x)$ defined on I such that

(i′) $$\alpha_{k,\nu}^*(\phi) = \left[\int_I x^{2\nu+1} | \Lambda_\nu^k \phi(x) |^2 dx\right]^{\frac{1}{2}}$$

exsts for each $k = 0, 1, 2, \ldots$

(ii′) $$(\Lambda_k^\nu \phi, \phi_n^*) = (\phi, \Lambda_\nu^{*k} \phi_\nu^*)$$

holds for each n and k.

As before, A_ν^* is a Fréchet space. $A_\nu^{*\prime}$ represents the dual space of A_ν^*.

Some properties related to these spaces are listed below

(a′) $D(I) \subset A_\nu^* \subset E(I)$. $E'(I)$ is a subspace of $A_\nu^{*\prime}$.

(b′) Note that $\phi_n(x)$, given by (4.5), is now in A_ν^*.

(c′) The operation $\phi \to \Lambda_\nu \phi$ is a continuous linear mapping of A_ν^* into itself. Hence the operation $f \to \Lambda_\nu^* f$ defined on $A_\nu^{*\prime}$ by

$$(\Lambda_\nu^* f, \phi) = (f, \Lambda_\nu \phi)$$

for any $\phi \in A_\nu^*$, is also a continuous linear mapping of $A_\nu^{*\prime}$ into itself.

Remark 1. Since $\{\phi_n^*\}$ is an orthogonal system on I with respect to the weight function $x^{-2\nu-1}$, verifying the same orthogonality condition (2.9), we propose to consider this other finite Hankel–Schwartz transform

$$(h_\nu^* f)(n) = F_\nu^*(n) = \int_0^a x^{-2\nu-1} \phi_n^*(x) f(x)\, dx = \int_0^a \mathcal{J}_\nu(j_n x) f(x)\, dx, \tag{4.11}$$

the inversion formular being given through

$$(h_\nu^{*-1} F_\nu^*)(x) = f(x) = \sum_{n=1}^{\infty} \frac{2F_\nu^*(n)}{j_n^2 a^{2\nu+4} \mathcal{J}_{\nu+1}^2(j_n x)}\, x^{2\nu+1} \mathcal{J}_\nu(j_n x) \tag{4.12}$$

By using a similar reasoning as in the proof of Theorem 2, we can establish

THEOREM 3. *Let* $\nu \geqq -\frac{1}{2}$. *If* $\phi \in A_\nu^*$, *then*

$$\phi = \sum_{n=1}^{\infty} \frac{2}{j_n^2 \mathcal{J}_{\nu+1}^2(j_n)} (\phi, \phi_\nu^*) \phi_n, \tag{4.13}$$

where the series converges in A_ν^*.

REMARK 2. Observe that

$$(h_\nu \phi)(x) = \phi_\nu(n) = (\phi, \phi_n^*), \quad \phi \in A_\nu^*, \tag{4.14}$$

is the finite Hankel-Schwartz transformation (3.12) acting on the space A_ν^*. Thus Theorem 3 can be interpreted as the inversion Theorem 1 for all testing function $\phi \in A_\nu^*$ and $a=1$. Analogously,

$$(h_\nu^* \phi)(x) = \phi_\nu^*(n) = (\phi, \phi_n), \quad \phi \in A_\nu, \tag{4.15}$$

can be considered as the finite transform (4.11) with $a=1$. Its inversion formula is given in the space A_ν by (4.12).

REMARK 3. Assume that $\nu \geqq -\frac{1}{2}$. Then A_ν may be identified with a subspace of $A_\nu^{*\prime}$, that is, $A_\nu \subset A^{*\prime}$. Indeed, every member $f \in A_\nu$ generates a regular distribution in $A_\nu^{*\prime}$ by

$$(f, \phi) = \int_I f(x) \phi(x) dx, \quad \phi \in A_\nu^*,$$

since

$$|(f, \phi)| \leqq \alpha_{0,\nu}(f) \cdot \alpha_{0,\nu}^*(\phi)$$

Furthermore, two members of A_ν which give rise to the same member of $A_\nu^{*\prime}$ must be identical.

In a similar way A_ν^* can be considered as a subspace of A_ν'.

5. Orthogonal series expansions and the finite generalized Hankel-Schwartz integral transformation

The main result of this section can be stated as follows

THEOREM 4. *Let* $\nu \geqq -\frac{1}{2}$. *Every member* $f \in A_\nu'$ *can be expanded into a generalized series of the form*

$$f = \sum_{n=1}^{\infty} \frac{2}{j_n^2 \mathcal{J}_{\nu+1}^2(j_n)} (f, \phi_n^*) \phi_n, \tag{5.1}$$

which converges in A_ν'.

Proof. By virtue of Theorem 2 it is inferred that

$$(f,\phi)=(f,\ \Sigma\frac{2}{j_n^2\mathcal{J}_{\nu+1}^2(j_n)}(\phi,\phi_n)\phi_n^*)=$$

$$=\Sigma\frac{2}{j_n^2\mathcal{J}_{\nu+1}^2(j_n)}(f,\phi_n^*)\overline{(\phi,\phi_n)}=\Sigma\frac{2}{j_n^2\mathcal{J}_{\nu+1}^2(j_n)}(f,\phi_n^*)(\phi_n,\phi),$$

for all $\phi\in A_\nu$. This implies that (5.1) truly converges in A_ν'.

Through an argument similar we can also assert

THEOREM 5. *Let* $\nu\geqq-\frac{1}{2}$. *If* $f\in A_\nu^{*\prime}$, *then*

$$f=\sum_{n=1}^{\infty}\frac{2}{j_n^2\mathcal{J}_{\nu+1}^2(j_n)}(f,\phi_n)\phi_n^*, \tag{5.2}$$

where the series converges in $A_\nu^{*\prime}$.

In view of (5.1) the finite Hankel-Schwartz transformation of the first kind of $f\in A_\nu'$ is defined by

$$(h_\nu f)(n)=F_\nu(n)=(f(x),\phi_n^*(x))=(f(x),x^{2\nu+1}\mathcal{J}_\nu(j_nx)) \tag{5.3}$$

for each $n=1, 2, \ldots$. Observe that this definition has a sense by virtue of note (b) in section 4. Its corresponding inversion formula is supplied by Theorem 4 and can be expressed as

$$(h_\nu^{-1}F_\nu)(x)=f(x)=\sum_{n=1}^{\infty}\frac{2F_\nu(n)}{j_n^2\mathcal{J}_{\nu+1}^2(j_n)}\mathcal{J}_\nu(j_nx) \tag{5.4}$$

We need merely invoke (4.10) to get

$$(\Delta_\nu^k f,\phi)=(f,\Delta_\nu^{*k}\phi)$$

for all $\phi\in A_\nu$ and $k=0, 1, 2\ldots$. If ϕ is replaced by ϕ_n^* and (4.7) is used, we yield

$$(\Delta_\nu^k f,x^{2\nu+1}\mathcal{J}_\nu(j_nx))=(f,(-j_n^2)^k x^{2\nu+1}\mathcal{J}_\nu(j_nx))$$

This formula may be rewritten in accordance with (5.3) as

$$h_\nu(\Delta_\nu^k f)=(-j_n^2)^k h_\nu f \tag{5.5}$$

for every $f\in A_\nu'$ and $k=0, 1, 2\ldots$.

REMARK 4. Theorem 5 suggests to introduce other variant of the finite Hankel-Schwartz transform of the first kind in the space $A_\nu^{*\prime}$ by means of

$$(h_\nu^* f)(n)=F_\nu^*(n)=(f,\phi_n)=(f,\mathcal{J}_\nu(j_nx)), \tag{5.6}$$

where $f\in A_\nu^{*\prime}$, for each $n=1, 2, \ldots$. The inversion formula is given through

$$(h_\nu^{*-1}F_\nu^*)(x)=f(x)=\sum_{n=1}^{\infty}\frac{2F_\nu^*(n)}{j_n{}^2\mathcal{J}_{\nu+1}^2(j_n)}x^{2\nu+1}\mathcal{J}_\nu(j_nx) \qquad (5.7)$$

REMARK 5. Since $A_\nu^*\subset A_\nu'$, our classical finite Hankel-Schwartz transform (4.14) is a special case of the generalized transformation (5.3) and Theorem 4 turns out to be an extension to distributions of Theorem 3. Similarly, as an immediate consequence of the inclusion $A_\nu\subset A_\nu^{*\prime}$, the classical finite transform (4.15) agrees with the generalized Hankel-Schwartz transformation (5.6), so that Theorem 5 appears now as the distributional version of Theorem 2.

REMARK 6. Let $\boldsymbol{N}$ be a linear differential operator and denote by $\boldsymbol{N}^*$ its adjoint operator. Zemanian [15, p. 264] investigated only the case $\boldsymbol{N}=\boldsymbol{N}^*$. However, the method developed here allows to takle more general problems (e. g., as is the case of our operators Λ_ν and Λ_ν^*) provided that, of course, both operators have the same eigenvalues and their respective systems of eigenfunctions verify also an identical orthogonality condition with respect to suitable weight functions.

6. Applications

To illustrate the applications of the finite Hankel-Schwartz transformation, we end this work by considering the following generalized Dirichlet problem:

Find the conventional solution $v(r, z)$ of the equation

$$\frac{\partial^2 v}{\partial r^2}+\frac{2\nu+1}{r}\ \frac{\partial v}{\partial r}+\frac{\partial^2 v}{\partial z^2}=0,\ \ 0<r<1,\ \ 0<z<\infty, \qquad (6.1)$$

satisfying the boundary conditions

(i) As $z\to 0+$, $v(r, z)\to f(r)\in A_\nu'$

(ii) As $z\to\infty$, $v(r, z)$ converges uniformly to zero on $0<r<1$.

(iii) As $r\to 1-$, $v(r, z)$ converges to zero on $c\leqq z<\infty$, for each $c<0$.

(iv) As $r\to 0+$, $v(r, z)=O(1)$ on $c\leqq z<\infty$.

Denote $V(n, z)=h_\nu[v(r, z)]$. According to (4.2), (6.1) becomes

$$\Lambda_{\nu, r}v+\frac{\partial^2 v}{\partial z^2}=0 \qquad (6.2)$$

By applying h_ν to (6.2) and making use of (5.5) we arrive at

$$-j_n^2 V(n, z) + \frac{\partial^2}{\partial z^2} V(n, z) = 0,$$

whose solution is

$$V(n, z) = F(n) e^{-j_n z}$$

because of boundary conditions (i) and (ii). Here $F(n) = h_\nu f$. We may now invoke our inversion formula (5.4) to get the solution

$$v(r, z) = \sum_{n=1}^{\infty} \frac{2F(n) e^{-j_n z}}{j_n^2 \mathcal{J}_{\nu+1}^2(j_n)} \mathcal{J}_\nu(j_n r) \tag{6.3}$$

Observe that the case in which $\nu=0$ is of special interest, since (6.1) coincides then with the Dirichlet problem for a semi-infinite cylinder considered by Zemanian. It is also worth noting that, when $\nu=0$, (6.3) yields directly the solution

$$v(r, z) = \sum_{n=1}^{\infty} \frac{2(f(y), yJ_0(j_n y))}{J_1^2(j_n)} e^{-j_n z} J_0(j_n r),$$

making unnecessary the previous changes of variables done in [16, p. 281].

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