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SOME LINEARITY OF CYCLIC ACTIONS ON S^4 WITH TWO FIXED POINTS

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1. Introduction

Suppose a group G acts smoothly and effectively on S^4 with the fixed point set $F(G, S^4)$ consisits of two points. A typical example of such actions is the unit sphere $S(V \oplus \boldsymbol{R})$ of an orthogonal representation $V \oplus \boldsymbol{R}$ of G where $\boldsymbol{R}$ is the trivial representation and V does not contain the trivial representation. Such actions are called linear actions. In transformation group theory we usually compare general actions with some well known actions such as linear actions. In this paper we consider cyclic group actions on S^4 with two isolated fixed points, and try to understand how similar they are to linear actions.

Before we state our main theorems let us study some basic materials related to the subject. Let t^s denote the following complex 1-dimensional irreducible representation of a cyclc group $\boldsymbol{Z}_n$ of order n : if $\boldsymbol{Z}_n = \{g^k | g = \exp 2\pi i/n,\ 0 \leq i \leq n-1\}$, then $g \cdot z = \exp 2\pi si/n \cdot z$ for $z \in t^s$, where the right hand side of the equation is the complex multiplication. It is an elementary fact from group representation theory that $\{t^0, t^1, \ldots, t^{n-1}\}$ is the set of all irreducible representations, see [Se]. As a real representation t^i is irreducible and isomorphic to t^{n-i} for $1 \leq i < \left[\frac{n}{2}\right]$. Here $\left[\frac{n}{2}\right]$ is the greatest integer less than or equal to $\frac{n}{2}$. If n is even, $t^{n/2}$ is not irreducible and isomorphic to $2\boldsymbol{R}_-$. Here $\boldsymbol{R}_-$ is the nontrivial real 1-dimensional representation where the generator $g = \exp 2\pi i/n$ acts on $x \in \boldsymbol{R}_-$ by $gx = -x$.

Note that if a group G acts smoothly on a smooth manifold M and if $x \in F(G, M)$, then the tangent space T_xM at x inherits a real representation structure of G by $\varphi : G \to GL(T_xM)$ with $\varphi(g)(v) = dg$

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(v) for $v \in T_xM$ and $g \in G$, where dg is the differential of the diffeomorphism $g: M \to M$. Such a real representation T_xM of G is called an isortopy representation of G at x. Suppose a cyclic group $G \in \boldsymbol{Z}_n$ acts linearly on S^4 with two fixed points x and y. Then it is easy to see that the fixed point set $F(H, S^4)$ for each subgroup H of G is an even dimensional sphere, and isotropy representations T_xS^4 and T_yS^4 of G at x and y are isomorphic.

Our main results are as follows:

THEOREM 1. *Suppose a cyclic group G of order n acts smoothly on S^4 with two fixed points. Then the fixed point set $F(H, S^4)$ for each subgroup H of G is an even dimensional sphere.*

For any representation V of G and a subgroup H of G the representation $\mathrm{res}_H V$ is the restriction of V to H, namely $\mathrm{res}_H V$ is the composition

$$H \hookrightarrow G \to GL(V).$$

THEOREM 2. *Suppose a cyclic group G of order n acts smoothly on S^4 with 2 fixed points $F(G, S^4) = \{x, y\}$. Let V and W be their isotropy representations of G at x and y, respectively. Let G_2 be the 2–Sylow subgroup of G. If $res_{G_2} V$ is isormophic to $res_{G_2} W$, then V and W are isomorphic.*

Suppose G acts smoothly on S^4 with two fixed points. Let $f: S^4 \to S^4$ be the Thom collapsing map around an invariant neighborhood of one fixed point x. Then the target sphere of the map f is the unit sphere $S(V \oplus \boldsymbol{R})$ where V is the isotropy representation of G at x. Then theorem 1 implies that the restriction $f^H: F(H, S^4) \to F(H, S^4)$ of f to the fixed point set $F(H, S^4)$ by each subgroup H of G is a homotopy equivalence. Thus f is a G homotopy equivalence. Hence we have the following corollary:

COROLLARY 3. *Any cyclic action on S^4 with two fixed points is G homotopy equivalent to a linear action.*

Two real representations V and W of a group G is said to be Smith equivalent if there exists a G homotopy sphere Σ with two fixed points x and y, and their isotropy representations $T_x \Sigma$ and $T_y \Sigma$ of G are isomorphic to V and W, respectively. Many interesting questions may be asked, and we refer the reader to [Pe], or [Sul] for them. The question we are interested in here is as follows:

QUESTION 1. What is the minimal dimension of non-isomorphic Smith equivalent representations?

Cappell and Shaneson [CS] showed that the minimal dimension for pseudo-free non-isomorphic Smith equivalent representations for cyclic groups of order $4k$ with $k \geq 2$ is 9. Pseudo-free representations of a cyclic group $\boldsymbol{Z}_n$ is of the form

$$V = \sum_{(i,n)=1} a_i t^i + \boldsymbol{R}_-$$

Namely, the action of $\boldsymbol{Z}_n$ on V is free except for one-dimensional subspace. Note that every pseudo-free representation of a cyclic group is of odd dimension.

Any Smith equivalent representations of cyclic groups of dimension less than or equal to 3 are isomorphic. This follows trivially for representations of dimension 1, by Atiyah-Bott G-signature theorem for dimension 2, and by the above Capplell and Shaneson's result for dimension 3 because they are pseudo-free. If Theorem 2 is true for arbitrary cyclic representations, then we can say that the minimal dimension of non-isomorphic Smith equivalent representations of any compact Lie group is greater than 4, because that the set of all element of finite order in the circle group is dense and that every compact Lie group can be covered by conjugations of its maximal torus. We thus conclude this section with the following question:

QUESTION 2. Is the minimal dimension of non-isomorphic Smith equivalent representation of any compact Lie group greater than 4?

2. Proof of main theorems

Before we prove our main theorems let us state some theorems which we need for our proofs.

THEOREM 4. [AB] *Let a compact Lie group act smoothly on a homotopy sphere Σ with two fixed points $F(G, \Sigma) = \{x, y\}$. If the action of G on $\Sigma - F(G, \Sigma)$ is free (such actions are called semi-free), then isotropy representations of G at x and y are isomorphic.*

THEOREM 5. [Su2] *Let G be a cyclic group of order $2d$ where d is odd. Let H be the index 2 subgroup. Suppose G acts on a homotopy sphere Σ with two fixed points $F(G, \Sigma)=\{x, y\}$. If H is an isotropy subgroup of either one of isotropy representations $T_x \Sigma$ and $T_y \Sigma$ of G, then $T_x \Sigma$ and $T_y \Sigma$ are isomorphic.*

We also need Atiyah–Bott G–signature theorem; however we only need the following special form of the case when fixed point sets consist of finite isolated points. Let $G=\{g^i \mid g=\exp 2\pi i/n,\ 0 \leq i \leq n-1\}$ be a cyclic group of oder n.

THEOREM 6. [AB] *Suppose G acts smoothly on a smooth even dimensional manifold M with finite isolated fixed points $F(G, M)=\{x_1, \ldots, x_n\}$. Then the G–signature $Sign(g, M)$ satisfies the following equation:*

$$Sign(g, M)=\sum_{x_k \in F(G,M)} \nu(x_k)$$

with $\nu(x_k)=\prod\left(\pm\frac{1+g^{ai}}{1-g^{ai}}\right)$ *where the isotropy representation* $T_{x_k}M=\sum t^{ai}$. *Here the sign $\pm$ depends on the choice of orientation on $T_{x_k}M$.*

Suppose a cyclic group $G=\boldsymbol{Z}_n$ acts smoothly and effectively on S^4 with two fixed points $F(G, S^4)=\{x, y\}$. Let V and W be isotropy representations T_xS^4 and T_yS^4 of G at x and y, respectively. The we may write

$$V=t^a+t^b$$
$$W=t^c+t^d$$

where $1 \leq a, b, c, d \leq \frac{n}{2}$, because $t^i \cong t^{n-i}$ as a real representation.

For a G space X and a point $x \in X$ its isotropy subgroup G_x is equal to $\{g \in G \mid gx=x\}$. Let Iso (X) denote the collection $\{G_x \mid x \in X\}$. For a representation t^s of $G=\boldsymbol{Z}_n$ it is easy to see that Iso $(t^s-0)=\boldsymbol{Z}_\alpha$ where g. c. d $(n, s)=\alpha$. Next lemma is essential for proofs of our main theorems.

LEMMA 7. *Let V and W be as above, and non isomorphic. Then Iso $(V)=Iso(W)$. Moreover if $Iso(V)=\{1, H_1, H_2, G\}$, then $H_1 \neq H_2$ and $g.c.d(|H_1|, |H_2|)=1$.*

Proof. Supose $H_1=H_2$. If $H_1 \neq 1$, then $V=2t^a$ where $(n, a) \neq 1$, and $\text{Iso}(t^a-0)=H_1$. thus $\dim F(H_1, S^4)=\dim F(H_1, V)=4$, which shows that $F(H_1, S^4)=S^4$. This means that the action of G is not effective. Thus $H_1=H_2=1$. This shows that the action of G on S^4 is semi-free, otherwise there exists a p-subgroup P of G for prime p such that $F(P, S^4) \neq \{x, y\}$, and then by Smith theory $F(P, S^4)$ is a mod p homology sphere of dimension greater than 0, which is impossible because $F(P, V)=F(P, W)=0$. Thus by Theorem 3 representations V and W are isomorphic, which contradicts to the hypothesis. Thus $H_1 \neq H_2$. If g. c. d$(|H_1|, |H_2|)=\alpha \neq 1$, then, because $F(\boldsymbol{Z}_\alpha. S^4)=S^4$ the action of G on S^4 is not effective. Hence g. c. d $(|H_1|, |H_2|)=1$. If $\mathrm{Iso}(W)=\{1, H_3, H_4, G\}$, the by the same arguement as above $H_3 \neq H_4$ and g. c. d$(|H_3|, |H_4|)=1$. Since dim $F(P, S^4)=$either 2 or 4 for any p-subgroup P of G by Smith theory

$F(P, S^4)=$mod p homology 2-sphere or mod p homology 4-sphere
$=S^2$ or S^4.

Thus $|H_1|$ must divide either $|H_3|$ or $|H_4|$, say $|H_3|$. By the same arguement for H_3 its order $|H_3|$ must divide $|H_1|$. Hence $H_1=H_3$. Similarly $H_2=H_4$.

We now prove our main theorems.

THEOREM 1. *Suppose a cyclic group G of order n acts smoothly on S^4 with two fixed points. Then the fixed point set $F(H, S^4)$ for each subgroup H of G is an even dimensional sphere.*

Proof. For any subgroup H of G its fixed point set $F(H, S^4)$ must contain $F(G, S^4)=\{x, y\}$. If $F(H, S^4) \neq F(G, S^4)$, then the component of $F(H, S^4)$ containing x must have the dimension either 2 or 4. If the dimension is 4, then $F(H, S^4)=S^4$ and we are done. If the dimension is 2, then by lemma 7 the component of $F(H, S^4)$ containing y must have the dimension 2. Then there is a p-subgroup P of H such that $F(P, S^4)$ is a mod p homology 2-sphere. Since $F(H, S^4) \subset F(P, S^4)$ the observation of their dimensions implies that $F(H, S^4)=F(P, S^4)=$mod p homology 2-sphere$=S^2$.

THEOREM 2. *Suppose a cyclic group G of order n acts smoothly on S^4 with two fixed points $F(G, S^4)=\{x, y\}$. Let V and W be isotropy representations of G at x and y, respectively. Let G_2 be the 2-Sylow subgroup of G. If res_{G2} V is isomorphic to $res_{G2}W$, then V and W are isomorphic.*

Proof. Let

$$V=t^a+t^b$$
$$W=t^c+t^d$$

for $1\leq a, b, c, d\leq\left[\frac{n}{2}\right]$. Let $(n, a)=(n, c)=\alpha$ and $(n, b)=(n, d)=\beta$. Then $H_1\cong \boldsymbol{Z}_\alpha$ and $H_2\cong \boldsymbol{Z}_\beta$. Without loss of generality we may assume that $\beta\neq 1$. As we have observed in the proof of Theorem 1 $F(H_2, S^4)=S^2$. Then G/H_2 acts on $F(H_2, S^4)=S^2$ with two fixed points $F(G/H_2, F(H_2, S^4))=F(G, S^4)=\{x, y\}$ and their isotropy representations of G/H_2 are $T_xS^2=t^{\bar{b}}$ and $T_yS^2=t^{\bar{d}}$ where $\bar{b}=b/\beta$ and $\bar{d}=d/\beta$. By Theorem 5

$$0=\operatorname{Sign}(\bar{g}, S^2)=\nu(x)+\nu(y)$$

where

$$\nu(x)=\pm\frac{1+\bar{g}^{\bar{b}}}{1-\bar{g}^{\bar{b}}} \text{ and } \nu(y)=\pm\frac{1+\bar{g}^{\bar{d}}}{1-\bar{g}^{\bar{d}}}$$

for $\bar{g}=\exp\frac{2\pi i}{n/\beta}$.

Case I. Both $\nu(x)$ and $v(y)$ are note zero. This is equivalent to saying that $b\neq\frac{n}{2}$ and $d\neq\frac{n}{2}$ when n is even. Then

$$\frac{\nu(x)}{\nu(y)}=\pm\left(\frac{1+\bar{g}^{\bar{b}}}{1-\bar{g}^{\bar{b}}}\right)\cdot\left(\frac{1-\bar{g}^{\bar{d}}}{1+\bar{g}^{\bar{d}}}\right)=\pm 1.$$

Then $\bar{b}\equiv\pm\bar{d}$ (mod n/β), which implies that $b/\beta\equiv\pm d/\beta$ (mod n/β). Thus $b\equiv\pm d$ (mod n). Since $1\leq b, d\leq\left[\frac{n}{2}\right]$, $b=d$. Apply theorem 6 again for G actions on S^4. Then

$$0=\operatorname{Sign}(g, S^4)=\nu(x)+\nu(y)$$

where

$$\nu(x)=\pm\frac{1+g^a}{1-g^a}\cdot\frac{1+g^b}{1-b^b} \text{ and } \nu(y)=\pm\frac{1+g^c}{1-g^c}\cdot\frac{1+g^d}{1-g^d}$$

for $g=\exp\frac{2\pi i}{n}$. Note that by Lemma 7 $a\neq\frac{n}{2}$ and $c\neq\frac{n}{2}$ if n even.

$$\frac{\nu(x)}{\nu(y)}=\pm\left(\frac{1+g^a}{1-g^a}\right)\cdot\left(\frac{1-g^c}{1+g^c}\right)=\pm 1$$

implies that $a\equiv\pm c$(mod n) since $1\leq a, c\leq\left[\frac{n}{2}\right]$ and $a=c$. Since $a=c$ and $b=d$ two representations V and W are isomorphic.

Case II. n is even and $b=d=\frac{n}{2}$. If n is not divisible by 4, then by Theorem 5, V is isomorphic to W^2. If n is divisible by 4, then $(n, a)=(n, c)=1$, otherwise the action of G on S^4 is not effective. Let H be the index 2 subgroup of G. Then $H\cong \boldsymbol{Z}_{n/2}$. Since $F(H, S^4)=S^2$ which is, in particular, connected $\mathrm{res}_H V=\mathrm{res}_H W$. Thus if $V\ncong W$, then

$$V=t^a+2\boldsymbol{R}_-$$
$$W=t^{a+b}+2\boldsymbol{R}_-=t^{n-a-b}+2\boldsymbol{R}_-=t^{b-a}+2\boldsymbol{R}_-.$$

Since $\mathrm{res}_{G2}\ V=\mathrm{res}_{G2}\ W$, $a\equiv a+b \pmod{|G_2|}$. Thus $b\equiv 0 \pmod{|G_2|}$, which is a contradiction. Therefore $V\cong W$.

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