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ON CONTACT *CR* SUBMANIFOLDS

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0. Introduction

The purpose of this paper is to study the differential geometric theory of submanifolds immersed in Sasakian manifolds. Throughout in this paper we assume that all submanifolds are tangent to the structure vector field of the ambient manifold. In the theory of submanifolds of Sasakian manifolds some notions that we have ever used do not make sense. For example, any invariant submanifold with parallel second fundamental form of a Sasakian manifold is totally geodesic. Therefore, we must define some new notions for submanifolds of Sasakian manifolds. We prove that these new notions correspond to that for submanifolds of Kaehlerian manifolds by using the method of Riemannian fiberations.

In §1 we state general formulas on submanifolds of Sasakian manifolds. We also give the definition of contact *CR* submanifolds and some basic results which later use. §2 is devoted to the study of totally contact geodesic submanifolds of Sasakian manifolds and the integrability conditions of an anti-invariant distribution and invariant distribution. In §3 we study the f-structures defined on a submanifold and give the conditions for submanifolds to be contact *CR* submanifolds. In §4 we prove a reduction theorem of codimension of an invariant submanifold of a Sasakian space form. §5 is devoted to the study of contact *CR* submanifolds with semi-flat normal connection and prove a classification theorem of contact *CR* submanifolds with semi-flat normal connection of a Sasakian space form. In the last §6 we prove some theorems which give the relations between submanifolds of Kaehlerian manifolds and submanifolds of Sasakian manifolds.

For the general theory of submanifolds of Riemannian manifolds, Kaehlerian manifolds and Sasakian manifolds see [28] and [29]. The

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fundamental theory of contact manifolds we refer to [16] and [29]. For the f-structures on manifolds see [16], [19] and [21].

1. Preliminaries

Let $\bar{M}$ be a $(2m+1)$-dimensional Sasakian manifold with structure tensors (ϕ, ξ, η, g). Then the structure tensors of M satisfy

$$\phi^2X=-X+\eta(X)\xi, \quad \phi\xi=0, \quad \eta(\phi X)=0, \quad \eta(\xi)=1,$$
$$g(\phi X, \phi Y)=g(X, Y)-\eta(X)\eta(Y), \quad \eta(X)=g(X, \xi)$$

for any vector fields X and Y on $\bar{M}$. We denote by $\bar{\nabla}$ the operator of covariant differentiation with respect to the metric g on $\bar{M}$. Then we have

$$\bar{\nabla}_X\xi=\phi X, \quad (\bar{\nabla}_X\phi)Y=-g(X, Y)\xi+\eta(Y)X=R(X, \xi)Y$$

for any vector fields X and Y on M, $\bar{R}$ denoting the Riemannian curvature tensor of $\bar{M}$.

Let M be an $(n+1)$-dimensional submanifold of $\bar{M}$ tangent to the structure vector field ξ. We denote by the same g the metric tensor field on M induced from that of $\bar{M}$. The induced connection on M will be denoted by ∇. Then the Gauss and Weingarten formulas are respectively given by

$$\bar{\nabla}_XY=\nabla_XY+B(X, Y) \quad \text{and} \quad \bar{\nabla}_XV=-A_VX+D_XV$$

for any vector fields X, Y tangent to M and any vector field V normal to M, where D denotes the operator of covariant differentiation with respect to the linear connection induced in the normal bundle $T(M)^{\perp}$. Both A and B are called the *second fundamental forms* of M and are related by

$$g(B(X, Y), V)=g(A_VX, Y).$$

The *mean curvature vector* μ of M is defined to be $\mu=(1/(n+1))\,TrB$. where TrB is the trace of B. If $\mu=0$, then M is said to be *minimal*. If the second fundamental form B satisfies $B(X, Y)=g(X, Y)\mu$ for all vector fields X and Y tangent to M, then M is said to be *totally umbilical*. In particular, if B vanishes identically on M, then M is called a *totally geodesic submanifold*.

We now prepare some basic formulas on M. For any vector field X tangent to M, we put

$$\phi X=PX+FX, \tag{1.1}$$

where PX is the tangential part of ϕX and FX the normal part of ϕX. Similarly, for any vector field V normal to M, we put

(1.2) $$\phi V = tV + fV,$$

whers tV is the tangential part of ϕV and fV the normal part of ϕV. We then have

(1.3) $$g(PX, Y) + g(X, PY) = 0, \quad g(fV, U) + g(V, fU) = 0,$$
$$g(FX, V) + g(X, tV) = 0.$$

Now, applying ϕ to (1.1) and (1.2), we respectively obtain

(1.4) $$P^2 = -I - tF + \eta \otimes \xi, \quad FP + fF = 0,$$
$$Pt + tf = 0, \qquad f^2 = -I - Ft.$$

Since $\phi\xi = P\xi = F\xi = 0$, we have

(1.5) $$P\xi = 0, \quad F\xi = 0.$$

We define the covariant derivatives of P, F, t and f by

$$(\nabla_X P) Y = \nabla_X (PY) - P\nabla_X Y, \quad (\nabla_X F) Y = D_X (FY) - F\nabla_X Y,$$
$$(\nabla_X t) V = \nabla_X (tV) - tD_X V, \quad (\nabla_X f) V = D_X (fV) - fD_X V,$$

respectively. For any vector field X tangent to M, we have

$$\bar{\nabla}_X \xi = \phi X = \nabla_X \xi + B(X, \xi),$$

from which, using (1.5),

(1.6) $$\nabla_X \xi = PX,$$

(1.7) $$B(X, \xi) = FX, \quad A_V \xi = -tV.$$

Furthermore we obtain

(1.8) $$(\nabla_X P) Y = A_{FY} X + tB(X, Y) - g(X, Y)\xi + \eta(Y) X,$$

(1.9) $$(\nabla_X F) Y = -B(X, PY) + fB(X, Y)$$

(1.10) $$(\nabla_X t) V = A_{fV} X - PA_V X,$$

(1.11) $$(\nabla_X f) V = -FA_V X - B(X, tV).$$

We denote by $\bar{M}^{2m+1}(c)$ a $(2m+1)$-dimensional Sasakian space form of constant ϕ-sectional curvature c. Then the curvature tensor R of an $(n+1)$-dimensional submanifold M of $\bar{M}^{2m+1}(c)$ is given by

(1.12) $$\begin{aligned} R(X, Y)Z = &\frac{1}{4}(c+3)[g(Y, Z)X - g(X, Z)Y] \\ &+ \frac{1}{4}(c-1)[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi + g(\phi Y, Z)\phi X \\ &- g(\phi X, Z)\phi Y + 2g(X, \phi Y)\phi Z] + A_{B(Y,Z)}X \\ &- A_{B(X,Z)}Y + (\nabla_Y B)(X, Z) - (\nabla_X B)(Y, Z). \end{aligned}$$

(1.13) $$\begin{aligned} R(X, Y)Z = &\frac{1}{4}(c+3)[g(Y, Z)X - g(X, Z)Y] \\ &+ \frac{1}{4}(c-1)[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \end{aligned}$$

$$+g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi$$
$$+g(PY,Z)PX-g(PX,Z)PY+2g(X,PY)PZ]$$
$$+A_{B(Y,Z)}X-A_{B(X,Z)}Y+(\nabla_Y B)(X,Z)$$
$$-(\nabla_X B)(Y,Z),$$

$$(\nabla_X B)(Y,Z)-(\nabla_Y B)(X,Z) = \frac{1}{4}(c-1)[g(PY,Z)FX-g(PX,Z)FY+2g(X,PY)FZ]. \tag{1.14}$$

Moreover, equation of Ricci is given by

$$g(R^{\perp}(X,Y)U,V)+g([A_V,A_U]X,Y) = \frac{1}{4}(c-1)[g(FY,U)g(FX,V)-g(FX,U)g(FY,V)+2g(X,PY)g(fU,V)] \tag{1.15}$$

Definition. Let M be a submanifold tangent to the structure vector field ξ isometrically immersed in a Sasakian manifold $\bar{M}$. Then M is called a *contact CR submanifold* of $\bar{M}$ if there exists a differentiable distribution $\mathcal{D}: x\rightarrow\mathcal{D}_x\subset T_x(M)$ on M satisfying the following conditions:

(i) $\mathcal{D}$ is invariant with respect to ϕ, i. e., $\phi\mathcal{D}_x\subset\mathcal{D}_x$ for each $x\in M$, and

(ii) the complementary orthogonal distribution $\mathcal{D}^{\perp}: x\rightarrow\mathcal{D}^{\perp}{}_x\subset T_x(M)$ is anti-invariant with retpect to ϕ, i. e., $\phi\mathcal{D}^{\perp}{}_x\subset T_x(M)$ for each $x\in M$,

where $T_x(M)$ and $T_x(M)^{\perp}$ denote the tangent space and normal space of M at $x\in M$, respectively.

We put $\dim M=2m+1$, $\dim M=n+1$, $\dim\mathcal{D}=h$, $\dim\mathcal{D}^{\perp}=q$ and $\operatorname{codim} M=2m-n=p$. If $q=0$, then a contact CR submanifold M is called an *invariant submanifold* of $\bar{M}$, and if $h=0$, then M is called an *anti-invariant submanifold* of $\bar{M}$ tangent to ξ. If $p=q$ and $\xi\in\mathcal{D}$, then a contact CR submanifold M is called a *generic submanifold* of M. If $h>1$ and $q>0$, then a contact CR submanifold M is said to be *proper* (*non-trivial*).

If M is an invariant submanifold of $\bar{M}$, then M is also a Sasakian manifold with respect to the induced stucture. On the other hand, we have $f=0$ and $t=0$, and $\phi X=PX$ for any vector field X tangent to M and $\phi V=fV$ for any vector field V normal to M.

We have the following lemmas (cf. [14], [29]).

LEMMA 1.1. *Let M be an invariant submanifold of a Sasakian manifold M. Then.*

(1.16) $$B(X,\xi)=0,\ A_V\xi=0,$$

(1.17) $$B(X,\phi Y)=B(\phi X,Y)=\phi B(X,Y)$$

(1.18) $$\phi A_V X+A_V\phi_x=0,\ A_V X=\phi A_V X.$$

LEMMA 1.2. *Let M be a contact CR submanifold of a Sasakian manifold $\overline{M}$. Then*

(1.19) $$FP=0,\ fF=0,\ tf=0,\ Pt=0,$$

(1.20) $$P^3+P=0,\ f^3+f=0.$$

In Lemma 1.2, (1.20) shows that P is an f-structure in M and f is an f-structure in the normal bundle of M.

LEMMA 1.3. *Let M be a contact CR submanifold of a Sasakian manifold $\overline{M}$. Then*

(1.21) $$A_{FX}Y-A_{FY}X=\eta(Y)X-\eta(X)Y \text{ for } X,\ Y\in\mathcal{D}^\perp.$$

For the integrability of the distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ of a contact CR submanifold M of a Sasakian manifold M, we have the following (cf. [28])

PROPOSITION 1.1. *Let M be an $(n+1)$-dimensional contact CR submanifold of a $(2m+1)$-dimensional Sasakian manifold $\overline{M}$. The distribution $\mathcal{D}^\perp$ is completely integrable and its maximal integral submanifold is a q-dimensional anti-invariant submanifold of $\overline{M}$ normal to ξ or a q-dimensional anti-invariant submanifold of $\overline{M}$ tangent to ξ, where $q=\dim\mathcal{D}^\perp$.*

PROPOSITION 1.2. *Let M be an $(n+1)$-dimensional contact CR submanifold of a $(2m+1)$-dimensional Sasakian manifold M. Then the disiribution $\mathcal{D}$ is completely integrable if and only if*

$$B(X,PY)=B(PX,Y)$$

for any vector fields $X,\ Y\in\mathcal{D}$, and then $\xi\in\mathcal{D}$. Moreover, the maximal integral submanifold of $\mathcal{D}$ is an h-dimensional invariant submanifold of $\overline{M}$, where $h=\dim\mathcal{D}$.

PROPOSITION 1.3. *Let M be a submanifold of a Sasakian manifold $\overline{M}$. If M is totally umbilical, then M is invariant and totally geodesic.*

Proof. By the assumption, we have $B(X, Y)g(X, Y)\mu$. Since $B(\xi, \xi)=0$, we obtain $\mu=0$. Thus M is totally geodesic. In this case, $FX=B(X, \xi)=0$. This shows that $\bar{M}$ is an invariant submanifold.

Moreover we have (cf. [13])

PROPOSITION 1.4. *If the second fundamental form of an invariant submanifold M of a Sasakian manifold $\bar{M}$ is parallel, then M is totally geodesic.*

2. Totally contact geodesic submanifolds

Let M be an $(n+1)$-dimensional Sasakian manifold $\bar{M}$. In view of Propositions 1.3 and 1.4, we need the following new concept.

DEFINITION. If the second fundamental form B of M satisfies

(2.1) $$(\nabla_X B)(Y, Z)=g(PX, Y)FZ+g(PX, Z)FY$$

for any vector fields $X, Y, Z\in\phi^2T(M)=T(M)-\{\xi\}$, then the second fundamental form B of M is said to be *contact parallel.*

DEFINITION. If the second fundamental form B of M satisfies

(2.2) $$B(X, Y)=\eta(X)FY+\eta(Y)FX$$

for any vector fields $X, Y\in T(M)$, then M is said to be *totally contact geodesic.*

DEFINITION. If the second fundamental form B of M is of the form

(2.3) $$B(X, Y)=[g(X, Y)-\eta(X)\eta(Y)]\alpha+\eta(X)FY+\eta(Y)FX$$

for any vector fields $X, Y\in T(M)$, α being a vector field normal to M, then M is said to be *totally contact umbilical.*

We easily see that M is totally contact geodesic if and only if $B(\phi^2X, \phi^2Y)=0$ for all $X, Y\in T(M)$, and M is totally contact umbilical if and only if $B(\phi^2X, \phi^2Y)=g(\phi^2X, \phi^2Y)\alpha$ for all $X, Y\in T(M)$.

From (1.5) and (2.2) we have

PROPOSITION 2.1. *If M is totally contact geodesic. then M is minimal.*

THEOREM 2.1. *Let M be a submanifold of $\bar{M}^{2m+1}(c)$ $(c\neq-3)$. If the second fundamental form B of M is contact parallel, then M is invariant or anti-invariant.*

Proof. From (2. 1) we obtain

$$(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = -g(PY, Z)FX + g(PX, Z)FY - 2g(X, PY)FZ$$

for any vector fields $X, Y, Z \in \phi^2 T(M)$. From this and (1. 14) we find

$$\frac{1}{4}(c+3)[g(PY, Z)FX - g(PX, Z)FY + 2g(X, PY)FZ] = 0.$$

Putting $Y=Z$ in the equation above, we obtain $g(X, PY)FY=0$. Thus we have $PY=0$ or $FY=0$. Consequently, M is anti-invariant or invariant.

If M is totally contact geodesic, (2. 2) implies

$$\begin{aligned}(\nabla_Z B)(X, Y) = {}& g(PZ, X)FY + g(PZ, Y)FX \\ & -\eta(X)B(Z, PY) - \eta(Y)B(Z, PX) \\ & +\eta(X)fB(Z, Y) + \eta(Y)fB(Z, X)\end{aligned}$$

for any vector fields X, Y and Z on M. Thus we have

$$(\nabla_Z B)(X, Y) = g(PZ, X)FY + g(PZ, Y)FX$$

for all $X, Y, Z \in \phi^2 T(M)$. Hence we have

PROPOSITION 2. 2. *If M is totally contact geodesic, then the second fundamental form B of M is contact parallel.*

From Theorem 2. 1 and Proposition 2. 2 we have (see [10])

THEOREM 2. 2. *Let M be a submanifold of $\bar{M}^{2m+1}(c)$ $(c \neq -3)$. If M is totally contact geodesic. then M is invariant or anti-invariant.*

Let M be an $(n+1)$-dimensional contact CR submanifold of a $(2m+1)$-dimensional Sasakian manifold $\bar{M}$. We suppose that M is totally contact geodesic in $\bar{M}$.

Let $X \in T(M)$ and $Y \in \mathcal{D}$. Then, by (1. 9) and (1. 18), we have

$$\begin{aligned}\phi\nabla_X Y &= P\nabla_X Y + F\nabla_X Y = P\nabla_X Y + \nabla_X FY - (\nabla_X F)Y \\ &= P\nabla_X Y - (\nabla_X F)Y = P\nabla_X Y + B(X, PY) - fB(X, Y) \\ &= P\nabla_X Y \in \mathcal{D}.\end{aligned}$$

Hence $\mathcal{D}$ is parallel and the maximal integral submanifold of $\mathcal{D}$ is totally geodesic in M.

When $\mathcal{D}$ is parallel, if $\mathcal{D}$ is normal to the structure vector field ξ, then

$$0 = g(\nabla_X Y, \xi) = -g(Y, PX)$$

for any $X, Y \in \mathcal{D}$. Thus we have $\dim \mathcal{D} = 0$. On the other hand, for any $X \in \mathcal{D}$, we have $\eta(X)\xi = \phi^2 X + X \in \mathcal{D}$, and hence $\xi \in \mathcal{D}$ or $\xi \in \mathcal{D}^\perp$.

$$0=-g(B(X, PZ), FY)+g(fB(X, Z), FY)=-g(A_{FY}X, PZ).$$

Thus we have $g(\nabla_X Y, P^2Z)=0$. Moreover, we have

$$g(\nabla_X Y, \xi)=-g(Y, \nabla_X \xi)=-g(Y, PX)=0,$$

from which $\nabla_X Y\in\mathcal{D}^\perp$. Hence the distribution $\mathcal{D}^\perp$ is parallel. This proves our theorem.

THEOREM 3.6. *Let M be a proper generic submanifold of a Sasakian space form $\bar{M}^{2m+1}(c)$. If P is contact parallel, then $c=-3$.*

Proof. In the proof of Theorem 3.4, we have $tB(X, PY)=0$ for any vector fields X and Y tangent to M. Thus we have

$$(\nabla_Z t)B(X, PY)+t(\nabla_Z B)(X, PY)+tB(X, (\nabla_Z P)Y)=0$$

for any vector field Z tangent to M, from which

$$\begin{aligned} t(\nabla_Z B)(X, PY)&=-(\nabla_Z t)B(X, PY)+B(PY, PZ)tFX \\ &=PA_{B(X, PY)}Z-A_{fB(X, PY)}Z+g(PY, PZ)tFX. \end{aligned}$$

Therefore, we have

$$\begin{aligned} t[(\nabla_Z B)(X, PY)-(\nabla_X B)(Z, PY)] &\\ =PA_{B(X, PY)}Z-A_{fB(X, PY)}Z-PA_{B(Z, PY)}X+A_{fB(Z, PY)}X &\\ +g(PY, PZ)tFX-g(PY, PX)tFZ.& \end{aligned}$$

Thus (1.14) implies

$$\begin{aligned} \frac{1}{4}(c+3)[g(PX, PY)tFZ-g(PZ, PY)tFX] &\\ =PA_{B(X, PY)}Z-A_{fB(X, PY)}Z-PA_{B(Z, PY)}X+A_{fB(Z, PY)}X.& \end{aligned}$$

We put $X=Y\in\mathcal{D}$ and $Z\in\mathcal{D}^\perp$. Then we obtain

Therefore, we see that the maximal integral submanifold of $\mathcal{D}$ is tangent to ξ.

LEMMA 2.1. *Let M be a contact CR submanifold of a Sasakian manifold $\bar{M}$. If M is totally contact geodesic in $\bar{M}$, then the distribution $\mathcal{D}$ is completely integrable and is tangent to ξ. Moreover, the maximal integral submanifold of $\mathcal{D}$ is an invariant submanifold of $\bar{M}$ and is a totally geodesic submanifold of M and $\bar{M}$.*

Proof. We prove that the maximal integral submanifold of $\mathcal{D}$ is totally geodesic in $\bar{M}$. Let X and Y be vector fields in $\mathcal{D}$. Then we have

$$\bar{\nabla}_X Y=\nabla_X Y+B(X, Y)=\nabla_X Y\in\mathcal{D},$$

because of $B(X, Y)=\eta(Y)FX+\eta(X)FY=0$. Therefore the maximal integral submanifold of $\mathcal{D}$ is totally geodesic in M.

Let $X\in T(M)$ and $Y\in\mathcal{D}^\perp$. Then $\eta(Y)=0$. Since M is totally contact geodesic in M, we have

$$\begin{aligned}\phi\nabla_X Y &= P\nabla_X Y+F\nabla_X Y=\nabla_X PY-(\nabla_X P)Y+F\nabla_X Y\\ &=-A_{FY}X-tB(X,Y)+g(X,Y)\xi+F\nabla_X Y\\ &=F\nabla_X Y\in T(M)^\perp.\end{aligned}$$

From this we see that $\nabla_X Y\in\mathcal{D}^\perp$. Consequently, $\mathcal{D}^\perp$ is parallel and the maximal integral submanifold of $\mathcal{D}^\perp$ is anti-invariant in $\overline{M}$ and is totally geodesic in $\overline{M}$. On the other hand, we have

$$\bar{\nabla}_X Y=\nabla_X Y+B(X,Y)=\nabla_X Y\in\mathcal{D}^\perp \text{ for } X, Y\in\mathcal{D}^\perp.$$

Therefore, the maximal integral submanifold of $\mathcal{D}^\perp$ is totally geodesic in M. From Lemmas 2.1 and 2.2 we have

Theorem 2.3. *Let M be a contact CR submanifold of a Sasakian manifold $\overline{M}$. If M is totally contact geodesic, then M is locally a Riemannian direct product $M^\top\times M^\perp$, where $M^\top$ is a totally geodesic invariant submanifold tangent to the structure vector field ξ of $\overline{M}$ and $M^\perp$ is a totally geodesic anti-invariant submanifold of $\overline{M}$.*

3. f-structures on submanifolds

Let M be a submanifold of a Sasakian manifold $\overline{M}$. If the endomorphism P on M is parallel, then (1.8) implies

$$0=A_{FPY}X+tB(X,PY)-g(X,PY)\xi,$$

from which

$$\begin{aligned}g(X,PY)&=g(A_{FPY}X,\xi)+g(tB(X,PY),\xi)\\ &=-g(X,tFPY)=g(X,PY)+g(X,P^3Y).\end{aligned}$$

Hence we have $P^3X=0$ for all $X\in T(M)$. This implies that $PX=0$ for all $X\in T(M)$. Therefore, we have (see [3], [18])

Proposition 3.1. *Let M be a submanifold of a Sasakian manifold $\overline{M}$. If the endomorphism P on M is parallel, then M is anti-invariant in $\overline{M}$.*

In view of Proposition 3.1 we need the following

Definition. If the endomorphism P on M is satisfies

$$(3.1)\qquad (\nabla_X P)Y=-g(PX,PX)\xi-\eta(Y)\phi^2X$$

for any vector fields X and Y tangent to M, then P is said to be *contact parallel*.

We use the following theorem (see [27]).

THEOREM 3.1. *In order for a submanifold M of a Sasakian manifold $\overline{M}$ to be a contact CR submanifold, it is necessary and sufficient that $FP=0$.*

THEOREM 3.2. *In order for a submanifold M of a Sasakian manifold $\overline{M}$ to be a contact CR submanifold it is necessary and sufficient that the endomorphism P is an f-structure on M, that is, $P^3+P=0$.*

Proof. If M is a contact CR submanifold, by (1.4), $P^3+P=0$. Conversely, suppose that $P^3+P=0$. Then (1.4) imples that $tFPX=0$ for all $X\in T(M)$. Thus we have, by (1.3),

$$0=g(tFPX, PX)=-g(FPX, FPX),$$

and hence $FPX=0$. From this and Theorem 3.1 we see that M is a contact CR submanifold.

THEOREM 3.3. *In order for a submanifold M of a Sasakian manifold $\overline{M}$ to be a contact CR submanifold it is necessary ad sufficient that f is an f-strucnure on M, that is, $f^3+f=0$.*

Proof. If M is a contact CR submanifold, by (1.4), $f^3+f=0$. We suppose that $f^3+f=0$. Then (1.4) implies that $fFtV=0$ for any vector field V normal to M. From (1.3) we obtain

$$0=g(fFtfV, V)=g(tfV, tfV),$$

from which $tf=0$. From this and (1.4) we have $PtV=0$, and hence

$$0=g(PtFPX, X)=g(FPX, FPX).$$

This gives $FP=0$. Therefore, Theorem 3.1 proves our assertion.

THEOREM 3.4. *Let M be a contact CR submanifold of a Sasakian manifold $\overline{M}$. If the f-structure P is contact parallel, then M is locally a Riemannian direct product $M^{\top}\times M^{\perp}$, where $M^{\top}$ is an invariant submanifold of $\overline{M}$ and M is an anti-invariant submanifold of M.*

Proof. From (1.8) and (3.1) we have

$$-g(PX, PY)\xi-\eta(Y)P^2X=A_{FY}X+tB(X, Y)-g(X, Y)\xi+\eta(Y)X,$$

from which

$$-g(PX, P^2Y)\xi=tB(X, PY)-g(X, PY)\xi.$$

Since P is an f-structure on M, it follows that

$$tB(X, PY)=0$$

for any vector fields X and Y tangent to M. Let $X\in T(M)$ and $Y\in\mathcal{D}$. From (1. 3) and (1. 9) we have

$$\begin{aligned} g(F\nabla_X Y, FZ) &= -g(((\nabla_X F)Y, FZ) \\ &= -g(fB(X, Y), F)Z+g(B(X, PY), FZ) \\ &= -g(tB(X, PY), Z=0 \end{aligned}$$

for any $Z\in T(M)$, from which, $F\nabla_X Y=0$. Hence the distribution $\mathcal{D}$ is parallel.

Let $X\in T(M)$ and $Y\in\mathcal{D}^{\perp}$. Then

$$P\nabla_X Y=-(\nabla_X P)Y=g(PX, PY)\xi-\eta(Y)P^2X=-\eta(Y)P^2X$$

Since $\mathcal{D}$ is parallel, we see that $\xi\in\mathcal{D}$ and $\mathcal{D}^{\perp}$ is normal to ξ. Thus $\eta(Y)=0$, and hence $P\nabla_X Y=0$. Therefore, we obtain $\nabla_X Y\in\mathcal{D}^{\perp}$. This means that $\mathcal{D}^{\perp}$ is parallel. From these considerations we have our theorem.

We next consider the normal bundle valued 1-form F on M. First of all, we prove

Proposition 3. 2. *Let M be a submanifold of a Sasakian manifold $\bar{M}$. Then F is parallel if and only if t is parallel.*

Proof. From (1. 9) and (1. 10) we obtain

$$\begin{aligned} (3.2)\quad g((\nabla_X F)Y, V) &= -g(B(X, PY), V)+g(fB(X, Y), V) \\ &= -g(A_V X, PY)-g(A_{fV}X, Y) \\ &= -g((\nabla_X t)V, Y). \end{aligned}$$

This proves our assertion.

Proposition 3. 3. *Let M be a submanifold of a Sasakian manifold $\bar{M}$. If F is parallel, then M is a contact CR submanifold of $\bar{M}$.*

Proof. From (1. 9) we have

$$0=(\nabla F)Y=-B(\xi, PY)+fB(\xi, Y).$$

Thus (1. 7) implies that $-FPY+fFY=0$ for any $Y\in M$, from which $FP-fF=0$. This, combining with (1. 4), gives $FP=0$. Therefore, Theorem 3. 1 proves our assertion.

Theorem 3. 5. *Let M be a submanifold of a Sasakian manifold $\bar{M}$. If F is parallel, then M is locally a Riemannian direct product $M^{\top}\times M^{\perp}$, where $M^{\top}$ is an invariant submanifold of M and $M^{\perp}$ is an anti-invariant submanifold of $\bar{M}$.*

Proof. Since F is parallel, we see that $\mathcal{D}$ is parallel. Let $X, Z \in T(M)$ and $Y \in \mathcal{D}^{\perp}$. Then

$$\begin{aligned} g(\nabla_X Y, P^2 X) &= -g(P\nabla_X Y, PZ) \\ &= g(A_{FY}X, PZ) + g(tB(X, Y), PZ) \\ &= g(A_{FY}X, PZ). \end{aligned}$$

On the other hand, (3.2) implies

$$\frac{1}{4}(c+3)g(PX, PY)g(Z, Z) = 0,$$

from which $c = -3$.

THEOREM 3.7. *Let M be a proper contact CR submanifold of a Sasakian space form $\bar{M}^{2m+1}(c)$. If F is parallel, then $c = -3$.*

Proof. From (1.9) we have

$$fB(X, Y) = B(X, PY) = B(PX, Y) \tag{3.3}$$

for any vector fields X and Y tangent to M. This means that

$$PA_V + A_V P = 0$$

for any vector field V normal to M. Thus we have

$$g(A_U PX, tV) = 0$$

for any vector field X tangent to M and any vector fields U and V normal to M. From this we obtain

$$g((\nabla_Y A)_U PX, tV) + g(A_U(\nabla_Y P)X, tV) + g(A_U PX, (\nabla_Y t)V) = 0.$$

Using (1.8) and (1.10), we have

$$\begin{aligned} & g((\nabla_Y A)_U PX, tV) + g(A_U A_{fX} Y, tV) + g(A_U tB(X, Y), tV) \\ & \quad - g(X, Y)g(A_U \xi, tV) + \eta(X)g(A_U Y, tV) + g(A_U PX, A_{fV} Y) \\ & \quad - g(A_U PX, PA_V Y) = 0, \end{aligned}$$

from which

$$\begin{aligned} g((\nabla_{PY} B)(PX, tV), U) &= g((\nabla_{PY} A)_U PX, tV) \\ &= -g(tB(X, PY), A_U tV) \\ &\quad - g(X, PY)g(tU, tV) \\ &\quad + g(A_U PX, A_V Y) + g(A_U X, A_{fV} P^2 Y). \end{aligned}$$

From this, combining equations of Codazzi (1.14), we obtain

$$\begin{aligned} & \frac{1}{2}(c+3)g(PX, Y)g(tU, tV) \\ & \qquad = -2g(A_U PX, A_V Y) + g(A_U Y, A_{fV} P^2 X) \\ & \qquad\quad - g(A_U X, A_{fV} P^2 Y). \end{aligned}$$

Consequently, we have

$$\frac{1}{2}(c+3)g(PX, PX)g(tV, tV)$$

$$=-2g(A_VPX, A_VPX)-2g(A_VPX, A_{fV}X).$$

We take V such that $V=FZ$ for some $Z\in\mathcal{D}^{\perp}{}_x$, then $fV=0$. Hence we have

$$\frac{1}{4}(c+3)g(PX, PX)g(Z, Z)=-g(A_{FZ}PX, A_{FZ}PX).$$

On the other hand, (3. 3) implies

$$g(A_{FZ}PX, Y)=g(B(PX, Y), FZ=g(fB(X, Y), FZ)=0,$$

and hence $A_{FZ}PX=0$. Thus we have $c=-3$.

4. Normal connection

Let M be an $(n+1)$-dimensional submanifold tangent to the sructure vector field ξ of a $(2m+1)$-dimensional Sasakian manifold $\overline{M}$.

DEFINITION. If the normal curvature tensor $R^{\perp}$ of M satisfies

$$R^{\perp}(\phi^2X, \phi^2Y)V=2g(PX, Y)fV$$

for any vector fields X and Y tangent to M and anty vector field V normal to M, then the normal connection of M is said to be ***contact flat***.

This condition is equivalent to

$$R^{\perp}(X, Y)V=2g(PX, Y)V$$

for any $X, Y\in\phi^2T(M)$ and $V\in T(M)^{\perp}$.

Let M be an $(n+1)$-dimensional coniact CR submanifold of $\overline{M}^{2m+1}(c)$ with contact flat normal connection. Then the Ricci equation (1. 15) implies

$$g([A_{fV}, A_V]PX, X)=\frac{1}{2}(c+3)g(PX, PX)g(fV, fV)$$

for any $X\in T(M)$ and $V\in T(M)^{\perp}$, from which

$$TrA_{fV}A_VP-TrA_VA_{fV}P=-\frac{1}{2}(c+3)\,TrP^2g(fV, fV).$$

If $PA_V=A_VP$ for all $V\in T_x(M)^{\perp}$, then $TrA_{fV}A_VP=TrA_VA_{fV}P$, and hence

(4. 1) $$(c+3)\,TrP^2g(fV, fV)=0.$$

THEOREM 4. 1. *Let M be a contact CR submanifold of a Sasakian space form $\overline{M}^{2m+1}(c)$ $(c\neq-3)$ with contact flat normal connection. If $PA_V=A_VP$ for any vector field V normal to M, then M is an anti-invariant submanifold or a generic submanifold of $\overline{M}^{2m+1}(c)$.*

Proof. From (4. 1) we see that $P=0$ or $f=0$. Thus M is anti-invariant or generic according as $P=0$ or $f=0$.

In the following we assume that M is a $(2n+1)$-dimensional invariant submanifold of a $(2m+1)$-dimensional Sasakian manifold M.

DEFINITION. If the normal curvature tensor $R^{\perp}$ of M satisfies

$$R^{\perp}(X, Y)V=\rho g(X, \phi Y)\phi V,$$

ρ being a function on M, for any vector fields X, Y tangent to M, then $R^{\perp}$ is said to be *ϕ-proportional.*

LEMMA 4.1. *Let M be a $(2n+1)$-dimensional $(n>1)$ invariant submanifold of a Sasakian manifold M. If $R^{\perp}$ is ϕ-proportional, then ρ is constant on M.*

Proof. We define $\nabla_X R^{\perp}$ by

$$(\nabla_X R^{\perp})(Y, Z)V=D_X(R^{\perp}(Y, Z)V)-R^{\perp}(\nabla_X Y, Z)V$$
$$-R^{\perp}(Y, \nabla_X Z)V-R^{\perp}(Y, Z)D_X V.$$

Thus, by a straightforward computation, we obtain

$$(\nabla_X R^{\perp})(Y, Z)+(\nabla_X R^{\perp})(Z, X)+(\nabla_Z R^{\perp})(X, Z)=0.$$

By the assumption we have

$$(\nabla_X R^{\perp})(Y, Z)V=(X\rho)g(Y, \phi Z)\phi V$$
$$+\rho g(Y, (\nabla_X\phi)Z)\phi V+\rho g(Y, \phi Z)(\nabla_X\phi)V,$$

from which

$$(\nabla_X R^{\perp})(Y, Z)V=(X\rho)g(Y, \phi Z)\phi V-g(X, Z)\eta(Y)\phi V$$
$$+\rho g(Y, X)\eta(Z)\phi V.$$

Thus we have

$$(X\rho)g(Y, \phi Z)+(Y\rho)g(Z, \phi X)+(Z\rho)g(X, \phi Y)=0.$$

Putting $Z=\phi Y$ in the equation above, we obtain

$$-(X\rho)g(Y, Y)+(X\rho)\eta(Y)\eta(Y)+(Y\rho)g(Y, X)$$
$$-(Y\rho)\eta(Y)\eta(X)+(\phi Y\rho)g(X, \phi Y)=0.$$

Then, assuming that Y is perpendicular to X, ϕX and ξ, we find

$$(X\rho)g(Y, Y)=0.$$

This shows that ρ is constant on M.

If the Ricci tensor S of a Sasakian manifold M is of the form

$$S(X, Y)=ag(X, Y)+b\eta(X)\eta(Y),$$

where a and b are functions on M, then M is said to be *η-Einstein* (cf. [29]).

THEOREM 4.2. *Let M be a $(2n+1)$-dimensional $(n>1)$ invariant submanifold of a Sasakian space form $\bar{M}^{2m+1}(c)$. If the normal curvature* *tensor of M is ϕ-proportional, then either M is totally geodesic or is an η-Einstein invariant submanifold of codimension 2 with scalar curvature $n(n(c+3)-2)$. The latter case occurs only when $c>-3$.*

Proof. From the Ricci equation (1.15) we have

$$\rho g(X, \phi Y) g(\phi U, V)+g([A_V, A_U]X, Y) = \frac{1}{2}(c-1) g(X, \phi Y) g(\phi U, V). \tag{4.2}$$

Putting $U=\phi V$ in this equation, we obtain

$$g(A_V{}^2X, \phi Y)=\frac{1}{4}[2\rho-(c-1)]g(X, \phi Y) g(V, V).$$

from which

$$g(A_VX, A_VX)=g(A_V\phi X, A_V\phi X) = \frac{1}{4}[2\rho-(c-1)]g(\phi X, \phi X) g(V, V). \tag{4.3}$$

If $2\rho=c-1$, then $A_V\phi X=0$, and hence $A_VX=0$ by Lemma 1.1. Therefore M is totally geodesic.

We next assume that $2\rho\neq c-1$ and $p=$codimension of $M>2$. Let $\{v_1, \ldots, v_p, \phi v_1, \ldots, \phi v_p\}$ be an orthonormal frame of $T_x(M)^{\perp}$. From (4.3) we have

$$A_aA_b+A_bA_a=0 \text{ for } a\neq b, \quad a, b=1, \ldots, p,$$

where we have put $A_a=A_{v_a}$.

On the other hand, from (4.2) we have

$$A_aA_b+A_bA_a=0.$$

Thus we have $A_aA_b=0$ for $a\neq b$. Consequently, (4.3) implies

$$0=g(A_bA_aX, A_bA_aX)=\frac{1}{4}[2\rho-(c-1)]g(\phi A_aX, \phi A_aX) g(V_b, V_b),$$

from which $\phi A_a=0$ and hence $2\rho=c-1$ by (4.3). This is a contradiction. Therefore, we must have $p=2$. Moreover, the Ricci tensor S of M is given by

$$S(X, Y)=\frac{1}{2}(n(c+3)+(c-1))g(X, Y) -\frac{1}{2}(n+1)(c-1)\eta(X)\eta(Y)-\Sigma g(A_a{}^2X, Y)$$

(cf. [29 ; p. 314]). We also have, by (4.3),

$$\Sigma g(A_a{}^2X, Y)=\frac{1}{4}[2\rho-(c-1)]g(\phi X, \phi Y)\Sigma g(v_a, v_a).$$

From these equations we see that M is η-Einstein. Thus our result reduces to the following theorem (cf. [29 ; p. 325]).

THEOREM 4.3. *Let M be a $(2n+1)$-dimensional η-Einstein invariant* *submanifold of codimension* 2 *of a Sasakian space form* $\overline{M}^{2m+1}(c)$. *If* $c \leqq -3$, *M is totally geodesic. If* $c > -3$, *M is either totally geodesic or the scalar curvature of M is* $n(n(c+3)-2)$,

If the normal connection of M is contact flat, then $\rho=-2$, and (4.3) implies

$$g(A_V X A_V X)=\frac{1}{4}(c+3)g(\phi X, \phi X)g(V, V).$$

It follows that $c \leqq -3$. Therefore, Theorem 4.2 reduces to

THEOREM 4.4. *Let M be an invariant submanifold of a Sasakian space form* $M^{2m+1}(c)$. *Then the normal connection of M is contact flat if and only if* $c=-3$, *and M is totally geodesic.*

On the other hand, we have ([11], [13])

THEOREM 4.5. *Let M be an invariant submanifold of a Sasakian space form* $\overline{M}^{2m+1}(c)$. *Then the normal connection of M is flat if and only if* $c=1$ *and M is totally geodesic.*

From the result of [9] (see also [7]) we have

THEOREM 4.6. *Let M be a* $(2n+1)$*-dimensional invariant submanifold of a Sasakian space form* $\overline{M}^{2m+1}(c)$. *If* $ImB=\{B(X, Y) : X, Y \in T(M)\}$ *is a* $2r$*-dimensional parallel subbundle of* $T(M)^{\perp}$, *then M is contained in a* $(2(n+r)+1)$*-dimensional totally geodesic invariant submanifold of* $\overline{M}^{2m+1}(c)$.

DEFINITION. If a normal vector field V of an invariant submanifold M satisfies

$$D_X V=\eta(X)DV=\eta(X)\phi V \text{ for all } X \in T(M)$$

or

$$D_Z V=0 \text{ for all } Z \in \phi^2 T(M),$$

then V is said to be *η-parallel.*

THEOREM 4.7. *Let M be a* $(2n+1)$*-dimensional invariant submanifold of a Sasakian space form* $\overline{M}^{2m+1}(c)$ $(c \neq -3)$. *Then M admits no η-parallel unit normal vector fields.*

Proof. Let V be an η-parallel unit normal vector field on M. Then, from (1.11), we have

$$D_X \phi V=D_X f V=(\nabla_X f)V=0,$$

because of $F=0$ and $t=0$. This shows that fV is also η-parallel. Let Q be the 2-dimensional subbundle of $T(M)^{\perp}$ generated by V and fV. Then Q is parallel, that is, Q is invariant under parallel translation. Moreover, we have

$$R^{\perp}(X, Y)V=2g(\phi X, Y)\phi V$$

for any $X, Y\in\phi T(M)$. Thus (1.15) implies

$$g([A_U, A_V]X, Y)=\frac{1}{2}(c+3)g(X, \phi Y)g(\phi V, U),$$

from which

$$2g(A_VX, A_VX)=-\frac{1}{2}(c+3)g(X, X)g(V, V).$$

Since $c\neq -3$, we have $c<-3$, and hence A_V is non-singular on $\phi^2T(M)$. On the other hand, we have $A_UA_V-A_VA_U=0$ because of $A_V\xi=0$. Thus we have $A_UA_V-A_VA_U=-\phi(A_UA_V+A_VA_U)=0$. Consequently, $A_UA_V=0$, and hence $A_U=0$ for any $U\in Q$. Therefore, we can see that M is contained in an $(2(n+1)+1)$-dimensional totally geodesic invariant submanifold $M^{2(n+1)+1}(c)\subset\overline{M}^{2m+1}(c)$. Since M is η-Einstein, by Theorem 4.3, M is totally geodesic in $M^{2(n+1)+1}(c)$ so that $A_V=0$. This contradicts the assumption that $c\neq -3$.

5. Semi-flat normal connection

Let M be an $(n+1)$-dimensional submanifold tangent to ξ of $\overline{M}^{2m+1}(c)$.

DEFINITION. M is said to have *semi-flat normal connection* if its normal curvature tensor $R^{\perp}$ satisfies

$$R^{\perp}(\phi^2X, \phi^2Y)V=\frac{1}{2}(c-1)g(X, PY)fV$$

for any $X, Y\in\phi^2T(M)$ and $V\in T(M)^{\perp}$.

In the following we see that M is an $(n+1)$-dimensional contact CR submanifold of $\overline{M}^{2m+1}(c)$ with semi-flat normal connection. Then (1.15) reduces to

$$(5.1)\qquad g([A_U, A_U]X, Y)=\frac{1}{4}(c-1)[g(FX, V)g(FY, U)-g(FX, U)g(FY, V)]$$

for any $X, Y\in\phi^2T_x(M)$ and $U, V\in T_x(M)$.

We now define an endomorphism

$$H_V : \phi^2 T_x(M) \to \phi^2 T_x(M), \quad x \in M$$

by

$$H_V X = -\phi^2 A_V X = A_V X - \eta(A_V X)\xi = A_V X + g(X, tV)\xi$$

for any $X \in \phi^2 T_x(M)$. Since $g(H_V X, \xi) = 0$, we see that $H_V X \in \phi^2 T_x(M)$. Thus the definition above is well defined. We can consider H_V as a symmetric (n, n)-matrix. We then have

$$g(A_U X, A_V Y) = g(H_U X, H_V Y) + g(FX, U) g(FY, V),$$
$$g(A_V X, A_U Y) = g(H_V X, H_U Y) + g(FY, U) g(FX, U),$$

for any $X, Y \in \phi^2 T_x(M)$ and $U, V \in T_x(M)^\perp$. Thus we obtain

$$(5.2) \quad g([A_V, A_U]X, Y) = g([H_V, H_U]X, Y) + g(FX, U) g(FY, V) - g(FX, V) g(FY, U).$$

Using (5.1) and (5.2), we have

$$(5.3) \quad g([H_V, H_U]X, Y) = \frac{1}{4}(c+3)[g(FX, V) g(FY, U) - g(FX, U) g(FY, V)].$$

LEMMA 5.1. *Let $X \in \mathcal{D}$ and $V \in fT(M)^\perp$. Then*

$$(5.4) \quad A_{fV} X = -A_V PX.$$

Proof. From (1.10) we have, for any $Y \in T(M)$,

$$g((\nabla_Y t) V, X) = g(\nabla_Y (tV) - tD_Y V, X) = 0,$$
$$g((\nabla_Y t) V, X) = g(A_{fV} Y, X) - g(PA_V Y, X) = g(A_{fV} X, Y) + g(A_V PX, Y).$$

Thus we have (5.4).

Here we notice that $A_{fV} X = H_{fV} X$ for any $X \in \phi^2 T(M)$ because of $g(A_{fV} X, \xi) = -g(X, tfV) = 0$, and hence

$$H_{fV} X = -H_V PX.$$

Let $X \in T(M)$ and $V \in fT(M)^\perp$. Then (5.3) implies

$$g([H_V, H_{fV}]PX, P^2 X) = 0,$$

from which

$$g(H_V H_{fV} PX, P^2 X) - g(H_{fV} H_V PX, P^2 X) = -g(H_V P^2 X, H_V P^2 X) - g(H_V PX, H_V PX) = 0.$$

Thus we have

$$(5.5) \quad H_V PX = 0, \text{ for } X \in T(M), \quad V \in fT(M)^\perp.$$

We also have, from (5.3),

$$g([H_{FX}, H_{fV}]PY, tU) = 0,$$

from which

$$g(H_{FX} H_{fV} PY, tU) - g(H_{fV} H_{FX} PY, tU)$$

$$=-g(H_{FX}PY, H_{fV}tU)=0$$

It follows that

(5.6) $$g(H_{FX}PY, H_{fV}tU)=0.$$

From (5.5) we can define an endomorphism

$$\bar{H}_V : tT_x(M)\rightarrow tT_x(M), \quad x\in M$$

by

$$\bar{H}_V Z=H_V Z$$

for any vectors $Z\in tT_x(M)$ and $V\in fT_x(M)^{\perp}$.

Let $h_1, \ldots, h_r$ be the distinct eigenvalues of H_V, and $E_1, \ldots, E_r$ the corresponding eigenspaces. Then

$$tT_x(M)=E_1\oplus\cdots\oplus E_r, \quad g(E_i, E_j)=0 \text{ for } i\neq j.$$

LEMMA 5.2. *Let M be a contact CR submanifold with semi-flat normal connection of $\bar{M}^{2m+1}(c)$ $(c\neq -3)$. Thus, for any V in $fT(M)^{\perp}$, $\bar{H}_V$ is proportional to the identity endomorphism.*

Proof. For any vectors $X, Y, Z\in tT_x(M)$ and $V\in T_x(M)^{\perp}$ from (5.3), we have

(5.7) $$g(H_V X, H_{FZ}Y)-g(H_V Y, H_{FZ}X).$$

If H_V is not proportional to the identity endomorphism, then $r>1$. Let $Z=X=Z_i\in E_i$, $Y=Z_i\in E_i$, for $i\neq j$. Then (5.7) implies

$$h_i g(Z_i, H_{Fi}Z_i)=h_i g(Z_i, H_{Fi}Z_i),$$

where we have put $Fi=FZ_i$ for any i to simplify the notation, from which

(5.8) $$g(H_{Fi}Z_i, Z_i)=0.$$

By linearity we have

(5.9) $$g(H_{Fi}E_i, E_i)=\{0\} \text{ for } i\neq j.$$

Putting $X=Z_i\in E_i$, $Y=Z_i\in E_i$ and $Z=Z_k\in E_k$ for $i\neq j$ in (5.7), we find

$$h_i g(H_{Fk}Z_i, Z_i)=h_i g(H_{Fk}Z_i, Z_i) \text{ for } i\neq j,$$

from which

(5.10) $$H_{Fk}E_i\subset PT_x(M)\oplus E_i.$$

Since we have $0=g(H_{Fk}X, H_V Z_i)=h_i g(H_{Fk}Z_i, X)$ for any $X\in PT_x(M)$, $Z_i\in E_i$ and $Z_k\in E_k$ by (4.7), we see that $H_{Fk}E_i\subset tT_x(M)^{\perp}$ if $h_i\neq 0$. Thus combining with (5.10), we find

(5.11) $$H_{Fi}E_i\subset E_i \text{ whenever } h_i\neq 0.$$

From (5.9) and (5.11) we have

(5.12) $$H_{Fi}E_i=0 \text{ if } j\neq i \text{ and } h_i\neq 0.$$

Since H_U has at least two distinct eigenvalues, we may assume that $h_1 \neq 0$. From (5.3) and (5.12) we obtain

$$(5.13)\quad 0=g(H_{F2}Z_1, H_{F2}Z_1)=\frac{1}{4}(c+3)g(Z_1, Z_1)g(Z_2, Z_2) + g(H_{F2}Z_2, H_{F1}Z_1).$$

On the other hand, we have

$$0=g(H_{Fi}Z_i, Z_j)=g(H_{Fj}Z_i, Z_i) \text{ for } i\neq j.$$

From this we get

$$(5.14)\qquad H_{Fi}Z_i \in PT_x(M)\oplus E_i.$$

Since $H_{F1}Z_1 \in E_1$ by (5.11), equations (5.13) and (5.14) give $c=-3$. This is a contradiction.

LEMMA 5.3. *Let M be a contact CR submanifold with semi-flat normal connection of $\overline{M}^{2m+1}(c)$ $(c\neq -3)$. Then for any $x\in M$, there is a unit normal vector $\mu \in fT_x(M)^\perp$ such that*

$$H_\mu X=0,\quad H_\mu Z=hZ,\quad H_V=0$$

for any vectors $X\in PT_x(M)$, $Z\in tT_x(M)^\perp$ and $V\in fT_x(M)^\perp$ such that $g(V,\mu)=0$.

Let $X, Y\in PT(M)$, $Z\in tT(M)^\perp$ and $V\in fT(M)^\perp$. Then equation of Codazzi implies

$$g((\nabla_X B)(Y,Z),V)-g((\nabla_Y B)(X,Z),V)=0.$$

On the other hand, we see that $A_V X=H_V X$ for $X\in PT(M)$, and hence $B(X,Z)=B(Y,Z)=0$ by (5.6) and Lemma 5.3. Thus we have

$$0=g(B([X,Y],Z),\mu)=g(AZ,[X,Y])$$
$$=g(HZ,[X,Y])=hg(Z,[X,Y]).$$

Thus if $h\neq 0$, then $g(Z,[X,Y])=0$ and hence $[X,Y]\in PT(M)\circ\{\xi\}=\mathcal{D}$. Since $[X,\xi]\in\mathcal{D}$, we see that $\mathcal{D}$ is completely integrable.

DEFINITION. A contact CR submanifold M is said to be *mixed totally contact geodesic* if

$$B(X,Z)=0 \text{ for } X\in PT(M),\quad Z\in\mathcal{D}^\perp.$$

This condition is equivalent to

$$B(PX,tV)=0 \text{ for } X\in T(M),\quad V\in T(M)^\perp.$$

DEFINITION. A contact CR submanifold M is said to be *mixed foliate* if it is mixed totally contact geodesic, and $\mathcal{D}=PT(M)\oplus\{\xi\}$ is completely integrable.

LEMMA 5.4. *Let M be an $(n+1)$-dimensional contact CR submanifold with semi-flat normal connection of $\bar{M}^{2m+1}(c)$ $(c\neq -3)$.*

(1) *Then h is constant, and for $X, Y\in\phi^2T(M)$ and $Z\in tT(M)^\perp$ we have*

$$(5.15)\qquad FR(X, Y)Z=B(X, P\nabla_Y Z)-B(Y, P\nabla_X Z)+h^2[g(Y, Z)FX-g(X, Z)FY],$$

$$(5.16)\qquad D_X FZ=F\nabla_X Z+hg(X, Z)f\mu.$$

(2) *If $h=0$, then M lies in a totally geodesic invariant submanifold $M^{n+1+p}(c)$ as a generic submanifold, where $p=\dim FT_x(M)$.*

(3) *If $h\neq 0$, then M is a contact CR submanifold with $fD\mu=0$, and M is mixed foliate.*

Proof. From (1.9) we have

$$(5.17)\qquad D_X FZ=F\nabla_X Z+fB(X, Z)\quad\text{for } X\in T(M),\ Z\in tT(M)^\perp.$$

By Lemma 5.3 we see that $fB(X, Z)=hg(X, Z)f\mu$ and hence (5.16). Thus (1.11) and (5.17) imply

$$D_X D_Y FZ=D_X(F\nabla_Y Z)+X(hg(Y, Z))f\mu-h^2g(Y, Z)FX+hg(Y, Z)fD_X\mu$$

by using $FA_\mu X=hFX$. From this and (1.9) we obtain

$$D_X D_Y FZ=fB(X, \nabla_Y Z)-B(X, P\nabla_Y Z)+F\nabla_X\nabla_Y Z+X(hg(Y, Z))f\mu-h^2g(Y, Z)FX+hg(Y, Z)fD_X\mu.$$

Since we have

$$\nabla_{[X,Y]}FZ=F\nabla_{[X,Y]}Z+hg([X, Y], Z)f\mu,$$

we obtain

$$\begin{aligned}R^\perp(X, Y)FZ=&FR(X, Y)Z+fB(X, \nabla_Y Z)-fB(Y, \nabla_X Z)-B(X, P\nabla_Y Z)\\&+B(Y, P\nabla_X Z)+[X(hg(Y, Z))-Y(hg(X, Z))]f\mu\\&-hg([X, Y], Z)f\mu-h^2[g(Y, Z)FX-g(X, Z)FY]\\&+h[g(Y, Z)fD_X\mu-g(X, Z)fD_Y\mu].\end{aligned}$$

On the other hand, we have

$$\begin{aligned}fB(X, \nabla_Y Z)&=g(fB(X, \nabla_Y Z), f\mu)f\mu\\&=g(A_\mu X, \nabla_Y Z)f\mu\\&=g(PA_\mu X, P\nabla_Y Z)f\mu+g(FA_\mu F\nabla_Y Z)f\mu\\&=hg(FX, F\nabla_Y Z)f\mu\\&=hg(X, \nabla_Y Z)f\mu-hg(PX, P\nabla_Y Z)f\mu,\end{aligned}$$

$$fB(Y, \nabla_X Z)=hg(Y, \nabla_Y Z)f\mu-hg(PY, P\nabla_X Z)f\mu.$$

Therefore, we have

$$(5.18)\qquad R^\perp(X, Y)FZ=FR(X, Y)Z-h[PX, P\nabla_Y Z)-g(PY, P\nabla_X Z)]f\mu$$

$$+[(Xh)g(Y,Z)-(Yh)g(X,Z)]f\mu$$
$$-B(X,P\nabla_Y Z)+B(Y,P\nabla_X Z)$$
$$-h^2[g(Y,Z)FX-g(X,Z)FY]$$
$$+h[g(Y,Z)fD_X\mu-g(X,Z)fD_Y\mu].$$

We also have

$$g(B(X,P\nabla_Y Z),fV)=g(A_{fV}X,P\nabla_Y Z)=g(X,H_{fV}P\nabla_Y Z)=0.$$

Hence we have

$$B(X,P\nabla_Y Z),\ B(Y,P\nabla_X Z)\in FT_x(M).$$

Since we have

$$R^{\perp}(X,Y)FZ=0 \text{ for } X,Y\in\phi^2T(M),$$

we obtain (5.15) and

(5.19) $$(Xh)g(Y,Z)-(Yh)g(Z,Z) =h[g(PX,P\nabla_Y Z)-g(PY,P\nabla_X Z)],$$

(5.20) $$h[g(Y,Z)fD_X\mu-g(X,Z)fD_Y\mu]=0$$

for any $X, Y, Z\in\phi^2T(M)$.

If M is invariant in $\bar{M}^{2m+1}(c)$, then (5.1) implies $g([A_V,A_V]X,Y)=0$ and hence

$$g(A_VA_VX,X)-g(A_VA_VX,X)=-2g(A_VX,A_VX)=0$$

by Lemme 1.1. Therefore, M is totally geodesic, and is a Sasakian space form of constant ϕ-sectional curvature c.

We next assume that M is not invariant. If $h=0$, then $B(X,Y)\in FT(M)$ for any $X, Y\in T(M)$ by $B(\xi,X)=FX$ and Lemma 5.3. We also have

$$0=g(B(X,Z),V)=g(\nabla_X Z,V)$$
$$=g(\nabla_X\phi Z,\phi V)=g(D_XFZ,fV)$$

for any $X\in T(M)$, $Z\in tT(M)^{\perp}$ and $V\in fT(M)^{\perp}$. This means that $FT(M)$ is a parallel normal subbundle. Because the first normal space $\operatorname{Im} B=\{B(X,Y): X, Y\in T(M)\}$ of M lie in $FT(M)$ Theorem 4.6 shows that M lies in a totally geodesic invariant submanifold $M^{n+1+p}(c)$ of $\bar{M}^{2m+1}(c)$. In this case, M is a generic submanifold of $M^{n+1+p}(c)$.

If $h\neq 0$, then $M'=\{x\in M: h(x)\neq 0\}$ is an open nonempty subset of M. We have seen that M' is a mixed foliate contact CR submanifold of $\bar{M}^{2m+1}(c)$ $(c\neq -3)$.

Here we need the following

LEMMA 5.5. *Let M be a mixed foliate proper contact CR submanifold of $\bar{M}^{2m+1}(c)$ $(c\neq -3)$. Then $c<-3$ and $p=dim\, FT_x(M)=dim\, tT_x(M)^{\perp}>1$.*

Proof. Let $X, Y \in PT(M)$ and $Z \in tT(M)$. Then

$$\begin{aligned}(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) &= -B([X, Y], Z) \\ &\quad -B(Y, \nabla_X Z) + B(X, \nabla_Y Z) \\ &= -2g(X, PY)FZ - B(Y, \nabla_X Z) \\ &\quad + B(X, \nabla_Y Z)\end{aligned}$$

because of $g([X, Y], \xi) = 2g(X, PY)$. If we take a vector field V normal to M such that $Z = tV$, $V \in FT(M)$, then $\nabla_X Z = -PA_V X + tD_X V$. Hence

$$\begin{aligned}(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) &= 2g(X, PY)V + B(Y, PA_V X) \\ &\quad -B(X, PA_V Y).\end{aligned}$$

From this and (1.14) we obtain

$$B(Y, PA_V X) - B(X, PA_V Y) = -\frac{1}{2}(c+3)g(PY, PY)g(V, V).$$

By the assumption, $c \neq -3$, and hence $c < -3$.

We next find that, if $p = 1$,

$$D^{\top}{}_X Z = 0 \text{ for } X \in \phi^2 T(M),\ \ Z \in tT(M)^{\perp},$$

where $D^{\top}$ denotes the normal connection of the maximal integral submanifold $M^{\top}$ of $\mathcal{D}$ in $\bar{M}^{2m+1}(c)$. Since $M^{\top}$ is an invariant submanifold, Theorem 4.7 gives a contradiction. Therefore, we have $p > 1$.

If $c > -3$, then M is anti-invariant by Lemma 5.5, and hence

$$(Xh)g(Y, Z) - (Yh)g(X, Z) = 0 \text{ for } X, Y \in \phi^2 T(M),\ \ Z \in tT(M)^{\perp},$$

by (5.19). Since $p > 1$, this implies $Xh = 0$ for all $X \in \phi^2 T(M)$. On the other hand, by (1.15) and (5.18), we have

$$\begin{aligned}g(R^{\perp}(\xi, Y)FZ, f\mu) &= -g([A_{f\mu}, A_{FZ}]\xi, Y) \\ &= -g(A_{FZ}A_{f\mu}\xi, Y) + g(A_{f\mu}A_{FZ}\xi, Y) \\ &= -g(A_{f\mu}tFZ, Y) = 0,\end{aligned}$$

$$g(R(\xi, Y)FZ, f\mu) = (\xi h)g(Y, Z) - (Yh)g(\xi, Z)$$

for any $Y \in T(M)$, $Z \in tT(M)^{\perp}$. Thus we have

$$(\xi h)g(Y, Z) - (Yh)g(\xi, Z) = 0 \text{ for } Y \in T(M),\ \ Z \in tT(M)^{\perp}.$$

Putting $Y = Z$ in this equation, we get $\xi h = 0$. Consequently, h is constant on M. Thus by (5.20) we get $fD_X\mu = 0$ for any $X \in \phi^2 T(M)$.

If $c < -3$, then Lemma 5.5 shows that $p > 1$. Thus for any unit vector $Z \in tT(M)^{\perp}$ there exists a unit vector $W \in T_x(M)^{\perp}$ so that $g(Z, W) = 0$. From (5.19) we find

$$Zh = 0 \text{ for } Z \in tT(M)^{\perp}.$$

Let X and Z be any unit vector fields in $PT(M)$ and $tT(M)^{\perp}$ respectively. Then (5.19) implies

$$Xh = hg(X, \nabla_Z Z),$$
$$g(X, \nabla_Z Z) = g(\phi X, \bar{\nabla}_Z \phi X) = -g(PX, A_{FZ} Z)$$
$$= -g(B(PX, Z), FZ) = 0,$$

from which $Xh=0$. Moreover, we see that $\xi h=0$ by (1.15) and (5.10). Consequently, $Xh=0$ for all $X \in T(M)$, and h is constant on M.

If $fD_X\mu=0$ for all $X \in \phi^2 T(M)$, from (5.18) we have

$$0 = g(R^{\perp}(\xi, Y)FY, fD_\xi\mu) = hg(Y, Y)g(fD_\xi\mu, fD_\xi\mu)$$

for any $Y \in PT(M)$. Thus we have $fD_\xi\mu=0$, and hence $fD_X\mu=0$ for all $X \in T(M)$.

THEOREM 5.1. *Let M be an $(n+1)$-dimensional contact CR submanifold of a Sasakian manifold space form $\bar{M}^{2m+1}(c)$ $(c \neq -3)$. Then M has semi-flat normal connection in $\bar{M}^{2m+1}(c)$ if and only if M is one of the following*:

(1) *a totally geodesic invariant submanifold $\bar{M}^{2m+1}(c)$,*

(2) *a flat anti-invariant submanifold of a totally geodesic invariant submanifold $M^{2n+1}(c)$ of $\bar{M}^{2m+1}(c)$,*

(3) *a proper generic submanifold with flat normal connection in a totally geodesic invariant submanifold $M^{n+1+q}(c)$ of $\bar{M}^{2m+1}(c)$,*

(4) *locally a Riemannian direct product $M^{\top} \times M_0$, where $M^{\top}$ is a space of positive constant sectional curvature and M_0 is a curve generated by ξ, immersed in a totally geodesic invariant submanifold $M^{2n+3}(c)$ of $\bar{M}^{2m+1}(c)$ with flat normal connection as an anti-invariant submanifold.*

Proof. If M is invariant in $\bar{M}^{2m+1}(c)$, then M is totally geodesic, and is a Sasakian space form $M^{n+1}(c)$.

Assume that M is not invariant in $\bar{M}^{2m+1}(c)$. Then $q=\dim FT_x(M)>0$, and there exists a unit normal vector field μ in Lemma 5.3.

If $h=0$ and M is anti-invariant, (5.15) shows that $FR(X, Y)Z=0$ for any $X, Y, Z \in \phi^2 T(M)$. On the other hand, we have $\nabla_X \xi = PX = 0$. Therefore, we obtain $R(X, Y)Z=0$ for any $X, Y, Z \in T(M)$. Thus M is flat.

If $h=0$ and M is neither invariant nor anti-invariant, then M lies in a totally geodesic invariant submanifold $M^{n+1+q}(c)$. In this case, M has flat normal connection.

If $h \neq 0$, Lemma 5.4 gives $D_X\mu \in F\mathcal{D}^{\perp}$. On the other hand, Lemma 5.3 also gives

$$(5.21)\qquad D_X f\mu=\bar{\nabla}_X\phi\mu=-\phi A_\mu X+\phi D_X\mu.$$

Since $A_\mu X\in\mathcal{D}^\perp$, $\phi D_X\mu\in T(M)$, we have $D\mu=0$. We have seen that $\mathcal{D}$ is integrable. Let M be a maximal integral submanifold of $\mathcal{D}$. We denote by $A^\top$ and $D^\top$ the second fundamental tensor and normal connection of $M^\top$ in $M^{2m+1}(c)$. Then we have

$$-A^\top{}_\mu X+D^\top{}_X\mu=\bar{\nabla}_X\mu=-A_\mu X+D_X\mu=0 \text{ for } X\in\phi^2T(M^\top).$$

This shows that $\mu|M^\top$ is η-parallel in the normal bundle of $M^\top$ in $\bar{M}^{2m+1}(c)$. This contradicts Theorem 4.7 unless M is anti-invariant in $\bar{M}^{2m+1}(c)$. If M is anti-invariant, (5.15) gives

$$FR(X, Y)Z=h^2[g(Y, Z)FX-g(X, Z)FY] \text{ for } X, Y, Z\in\phi^2T(M).$$

Thus we have

$$R(X, Y)Z=h^2[g(Y, Z)X-g(X, Z)Y] \text{ for } X, Y, Z\in\phi^2T(M).$$

Consequently, we get

$$R(X, Y)Z=h^2[g(Y, Z)X-g(X, Z)Y+\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X$$
$$+\eta(Y)g(X, Z)\xi-\eta(X)g(Y, Z)\xi]$$

for any $X, Y, Z\in T(M)$. From this and a theorem of [19] we see that M is locally $M^\top\times M_0$. Moreover, M has flat normal connection.

From (5.21) and $D\mu=0$ we have

$$D_X\mu=-\phi A_\mu X\in\phi\mathcal{D}^\perp \text{ for } X\in T(M).$$

Therefore, we see that $\phi\mathcal{D}^\perp\oplus\mathrm{Span}\{\mu, \phi\mu\}$ is a parallel normal subbundle, and $\mathrm{Im}\, B\subset\phi\mathcal{D}^\perp\oplus\mathrm{Span}\{\mu, \phi\mu\}$. From these we conclude that M lies in a totally geodesic invariant submanifold $M^{2n+3}(c)$ of $\bar{M}^{2m+1}(c)$ as an anti-invariant submanifold with flat normal connection.

The converse of this is trivial.

6. Fibering of submanifolds

Let $\bar{M}$ be a $(2m+1)$-dimensional Sasakian manifold with structure tensors (ϕ, ξ, η, g) such that there exists a fibering $\bar{\pi}: \bar{M}\to\bar{M}/\xi=\bar{N}$, where N denotes the set of orbits of ξ and is a real $2m$-dimensional Kaehlerian manifold. We denote by (J, G) the Kaehlerian structure of $\bar{N}$ and by the horizontal lift with respect to the connection η. Then we have

$$(JX)^*=\phi X^*,\quad g(X^*, Y^*)=G(X, Y)$$

for any vector fields X and Y on $\bar{N}$, where we write $G(X, Y)\cdot\pi$ by $G(X, Y)$ to simplify the notation. We denote by $\bar{\nabla}$ (resp. $\bar{\nabla}'$) the operator of covariant differentiation with respect to g (resp. G). Then we obtain

$$(\bar{\nabla}'_X Y)^* = -\phi^2\bar{\nabla}_{X*}Y^* = \bar{\nabla}_{X*}Y^* + g(\phi X^*, Y^*)\xi$$

Let M be an $(n+1)$-dimensional submanifold tangent to the structure vector field ξ of $\bar{M}$ and N be an n-dimensional submanifold of $\bar{N}$. In the sequel, we assume that there exists a fibration $\pi : M \rightarrow N$ such that the diagram

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[d, "\pi"'] & \bar{M} \arrow[d, "\bar{\pi}"] \\ N \arrow[r, "i'"] & \bar{N} \end{tikzcd}$$

commutes and the immersion i is a diffeomorphism on the fibres. We denote by the same g and G the induced metric tensor fields on M and N respectively. Let ∇ (resp. ∇') be the operator of covariant differentiation in M (resp. N). We denote by B (resp. B') the second fundamental form of the immersion i (resp. i') and by A (resp. A') the associate second fundamental tensor of B (resp. B'). Let X and Y be vector fields tangent to M. Then we have

$$\bar{\nabla}'_X Y = \nabla'_X Y + B(X, Y) \text{ and } \bar{\nabla}_{X*}Y^* = \nabla_{X*}Y^* + B(X^*, Y^*).$$

Therefore we obtain

$$(6.1)\qquad (\nabla'_X Y)^* = -\phi^2\nabla_{X*}Y^* = \nabla_{X*}Y^* + g(\phi X^*, Y^*)\xi,$$

$$(6.2)\qquad (B'(X, Y))^* = B(X^*, Y^*).$$

Let D and D' be the operators of covariant differentiation with respect to the linear connection induced in the normal bundles of M and N respectively. For any vector field X tangent to N and vector field V normal to N, we obtain

$$\bar{\nabla}'_X V = -A'_X V + D'_X Y \text{ and } \bar{\nabla}_{X*}V^* = -A_{X*}Y^* + D_{X*}Y^*,$$

from which

$$(6.3)\qquad (A'_V X)^* = -\phi^2 A_{X*}Y^* = A_{X*}Y^* - g(A_{X*}Y^*, \xi)\xi,$$

$$(6.4)\qquad (D'_X Y)^* = D_{X*}Y^*.$$

For any vector field X tangent to M we put $\phi X = PX + FX$ where PX is the tangential part of ϕX and FX the normal part of ϕX. Similarly, for any vector field V normal to M, we put $\phi V = tV + fV$, where tV is the tangential part of ϕV and fV the normal part of ϕV. We can define the operators P', F', t' and f' on N corresponding respectively to P, F, t and f. Then we have

$$(P'X)^*=PX^*, \quad (F'X)^*=FX^*$$

for any vector field X tangent to N. We also have

$$(t'V)^*=tV^*, \quad (f'V)^*=fV^*$$

for any vector field V normal to N. Thus we have (cf. [28])

PROPOSITION 6.1. (1) *M is a contact CR submanifold of $\overline{M}$ if and only if N is a CR submanifold of $\overline{N}$;*

(2) *M is a generic submanifold of $\overline{M}$ if and only if N is a generic submanifold of $\overline{N}$;*

(3) *M is an anti-invariant submanifold of $\overline{M}$ tangent to ξ if and only if N is anti-invariant submanifold of $\overline{N}$;*

(4) *M is an invariant submanifold of $\overline{M}$ if and only if N is an invariant submanifold (a complex submanifold) of $\overline{N}$.*

We compute the relation between covariant differentiation of the second fundamental forms B and B'. From (6.1), (6.2) and (6.4) we obtain

$$(6.5) \qquad (\nabla_{X*}B)(Y^*, Z^*)=[(\nabla_X B')(Y, Z)+G(Y, P'X)F'Z + G(Z, P'X)F'Y]^*.$$

Moreover, we have

$$\begin{aligned}(\nabla_{X*}B)(Y^*, \xi) &= D_{X*}(FY^*)-F\nabla_{X*}Y^*-B(Y^*, PX^*)\\ &=(\nabla_{X*}F)Y^*-B(Y^*, PX^*)\\ &=fB(X^*, Y^*)-B(X^*, PY^*)-B(Y^*, PX^*),\end{aligned}$$

from which

$$(6.6) \quad (\nabla_{X*}B)(Y^*, \xi)=[f'B'(X, Y)-B'(X, P'Y)-B'(Y, P'X)]^*.$$

In the next place, we consider the normal connections of M and N. We denote by $R^\perp$ and $K^\perp$ the curvature tensors of the normal bundles of M and N respectively. We give the relation of $R^\perp$ and $K^\perp$. Let X and Y be vector fields tangent to N and U and V be vector fields normal to N. Then (6.4) implies

$$(D'_X D'_Y V)^*=D_{X*}D_{Y*}V^*, \quad (D'_Y D'_X V)^*=D_{Y*}D_{X*}V^*.$$

Since $[X, Y]^*=[X^*, Y^*]+2g(PX^*, Y^*)D_\xi V^*$, we find

$$(D'_{[X,Y]}V)^*=D_{[X*,Y*]}V^*+2g(PX^*, Y^*)D_\xi V^*.$$

From these equations we have

$$(6.7) \qquad \begin{aligned}&g(R^\perp(X^*, Y^*)V^*, U^*)\\ &\quad =[G(K^\perp(X, Y)U, V)+2G(P'X, Y)G(f'V, U)]^*.\end{aligned}$$

On the other hand, the Ricci equation of M implies

$$\begin{aligned}g(R^\perp(X^*, \xi)V^*, U^*) &= g([A_{U*}, A_{V*}]X^*, \xi)\\ &=g(A_{U*}X^*, tV^*)-g(A_{V*}X^*, tU^*)\end{aligned}$$

$$=-[G(F'A'_UX, V)+G(B'(X, t'U), V)]^*.$$

Thus we have

$$g([A_{U*}, A_{V*}]X^*, \xi)=G((\nabla_X f')V, U)^*, \tag{6.8}$$

from which

$$gR^{\perp}(X^*, \xi)V^*, U^*)=G((\nabla_X f')V, U)^*. \tag{6.9}$$

In the following we study the relations between submanifolds M tangen t to the structure vector field ξ of a Sasakian manifold $\overline{M}$ and submanifolds N of a Kaehlerian manifold $\overline{N}$.

From (2.1) and (6.5) we have

THEOREM 6.1. *The second fundamental form B of M is contact parallel if and only if the second fundametal form B' of N is parallel.*

From (2.2) and (6.2) we obtain

THEOREM 6.2. *M is totally contact geodesic if and only if N is totally geodesic.*

THEOREM 6.3. *M is totally contact umbilical if and only if N is totally umbilical.*

REMARK. About theorems for submanifolds of Kaehlerian manifolds which have parallel second fundamental form, or are totally geodesic or totally umbilical, see [6].

THEOREM 6.4. *The endomorphism P on M is contact parallel if and only if P' on N is parallel.*

Proof. For any $X, Y\in T(N)$ we have

$$\begin{aligned}((\nabla'_X P')Y)^* &= (\nabla'_X(P'Y))^*-(P'\nabla'_X Y)^*\\ &= -\phi^2\nabla_{X*}(P'Y)^*+P\phi^2\nabla_{X*}Y^*\\ &= -\phi^2(\nabla_{X*}P)Y^*\\ &= (\nabla_{X*}P)Y^*+g(PX^*, PY^*)\xi.\end{aligned}$$

From this we see that P' is parallel if and only if

$$(\nabla_X P)\phi^2 Y=-g(P\phi^2X, P\phi^2Y)\xi=-g(PX, PY)\xi$$

for any $X, Y\in T(M)$. Therefore, P' is parallel if and only if (3.1) holds. This proves our theorem.

THEOREM 6.5. *In order for F' on N to be parallel, it is necessary and sufficient that F on M satisfies*

$$(\nabla_X F)Y=-\eta(X)FPY+\eta(X)fFY-\eta(Y)FPX$$

for any $X, Y \in T(M)$.

Proof. For any $X, Y \in T(N)$ we have

$$\begin{aligned}((\nabla'_X F')Y)^* &= (\nabla'_X(F'Y))^* - (F'\nabla'_X Y)^* \\ &= -\phi^2\nabla_{X*}(F'Y)^* + F\phi^2\nabla_{X*}Y^* \\ &= (\nabla_{X*}Y)Y^*.\end{aligned}$$

Thus F' is parallel if and only if $(\nabla_X F)Y=0$ for all $X, Y \in \phi^2 T(M)$. This is equivalent to our equation by (1.9).

COROLLARY. *If* M *is a contact* CR *submanifold of* $\overline{M}$, *then* F *is parallel if and only if* F' *is parallel.*

THEOREM 6.6. *In order for* t' *on* N *to be parallel, it is necessary and sufficient that* t *on* M *satisfies*

$$(\nabla_X t)V = \eta(X)PtV - \eta(X)tfV$$

for any $X \in T(M)$ *and* $V \in T(M)^\perp$.

Proof. For any $X \in T(N)$, $V \in T(N)^\perp$, we have

$$((\nabla'_X t')V)^* = (\nabla_{X*}t)V^*.$$

Thus t' is parallel if and only if $(\nabla_X t)V=0$ for any $X \in \phi^2 T(M)$ and $V \in T(M)^\perp$. From (1.10) we see that this is equivalent to our condition.

COROLLARY. *If* M *is a contact* CR *submanifold of* $\overline{M}$, *then* t *is parallel if and only if* t' *is parallel.*

THEOREM 6.7. f *is parallel if and only if* f' *is parallel.*

Proof. For any $X \in T(N)$, $V \in T(N)$, we have

$$((\nabla'_X f')V)^* = (\nabla_{X*}f)V^*.$$

Therefore, f' is parallel if and only if $(\nabla_X f)V=0$ for any $X \in \phi^2 T(M)$ and $V \in T(M)$. Thus f' is parallel if and only if

$$(\nabla_X f)V = \eta(X)(\nabla_\xi f)V$$

for any $X \in T(M)$ and $V \in T(M)$. On the other hand, (1.11) gives

$$(\nabla_\xi f)V = -FA_V\xi - B(\xi, tV) = 0$$

by (1.7). Consequently, we have our assertion.

From (6.7) we have

THEOREM 6.8. *The normal connection of* N *is flat if and only if the normal connection of* M *is contact flat.*

From (6.7) and (6.9) we have

THEOREM 6.9. *Let M be an invariant submanifold of a Sasakian manifold M. Then the normal curvature tensor $R^{\perp}$ of M is ϕ-proportional if and only if the normal curvature tensor $K^{\perp}$ of N is J-proportional.*

REMARK. For invariant submanifold of a Kaehlerian space form with J-proportional normal curvature tensor, see [8].

THEOREM 6.10. *Let M be a submanifold of a Sasakian space form of constant ϕ-sectional curvature, and N be a submanifold of a Kaehlerian space form of constant holomorphic sectional curvature $c+3$. Then the normal connection of M is semi-flat if and only if the normal connection of N is semi-flat.*

REMARK. For CR submanifolds with semi-flat normal connection of Kaehlerian space form, see [4], [23], [28].

We also have, by (6.7) and (6.9).

THEOREM 6.11. *The normal connection of M is flat if and only if*

$$K^{\perp}(X, Y)=2G(X, P'Y)f' \quad \text{and} \quad \nabla f'=0$$

for any $X, Y\in T(N)$.

REMARK. For submanifolds of a Kaehlerian space form with the conditions in the above theorem, see [26].

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