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UNIQUENESS OF L^p SOLUTIONS FOR THE LAPLACE EQUATION AND THE HEAT EQUATION ON MINIMAL SUBMANIFOLDS

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1. Introduction

Consider the equation for harmonic functions

(1.1) $$\Delta f \equiv 0$$

defined on a complete C^∞ Riemannian manifold. It is well-known that nonconstant harmonic functions do not exist on compact manifold. On the other hand there always exist $2n+1$ harmonic functions to embed it as a closed submanifold of $\boldsymbol{R}^{2n+1}$ [3]. Hence it is necessary to restrict a function to lie in a suitable function space, to study harmonic functions on noncompact Riemannian manifold. Some of the most natural spaces are those consisting of functions on M, denoted by $L^p(M)$.

The uniqueness of L^p solutions of (1.1) means that if $f \in L^p(M)$ and f satisfies (1.1), then f must be identically constant. For $p \in (0, \infty]$, we say that a manifold satisfies property H_p if every L^p harmonic function on M is constant. We say that M satisfies property S_p if every nonnegative L^p subharmonic function on M is constant. Note that since the absolute value of a harmonic function is subharmonic, M satisfies S_p implies it satisfies H_p.

In [10], Yau proved that every nonnegative L^p subharmonic function on any complete noncompact manifold is constant for $1 < p < \infty$. Because of this we must concentrate our attention on positive or L^p ($0 < p \leq 1$, $p = \infty$) harmonic functions. Yau [9] proved that a complete manifold with nonnegative Ricci curvature does not admit nonconstant positive harmonic function. Anderson and Schoen [1] showed and gave a through understanding of the existence of bounded harmonic functions

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on a simply connected manifold with curvature bounded by two negative constants. An unpublished example of L. O. Chung showed that Yau's Theorem is false for $p=1$. Thus to deal with the L^1 solution, the additional hypothesis on the curvature of the manifold would be necessary. In [6], Li and Schoen established the conditions on the curvature on M to ensure the property S_p for $(0, 1]$ and provided examples of manifolds which possess nonconstant L^p harmonic functions for $p\in(0, 1]$.

For the heat equation, in [5], Li proved that any L^p solution of the heat equation is uniquely determined by its initial data in $L^p(M)$ for $1<p<\infty$. But we would like to point out that in general L^1 and L^∞ uniqueness does not hold. Actually Azencott constructed a counter example to the L^1 uniqueness of the heat equation [5]. Also Li [4], [5], gave optimal curvature condition for a complete Riemannian manifold to have unique bounded or L^1 solution for the heat equation with prescribed initial data.

In this paper, we prove the uniqueness of solutions of the Laplace equation and the heat equation by replacing the curvature condition by another geometric condition on M.

2. The Laplace equation

For minimal submanifolds, we have the following theorem.

THEOREM 1. *Let M be an n-dimensional complete noncompact minimal submanifold of N, a complete simply connected Riemannian manifold with nonpositive sectional curvature. Then M satisfies S_p for $p\in(0, \infty)$. In particular, M satisfies H_p for $p\in(0, \infty)$.*

Proof. Since M satisfies S_p for $p\in(1, \infty)$ [10], we suppose $p\in(0, 2]$ and suppose g is a nonnegative L^p subharmonic function on M. For a point $x\in M$, the geodesic ball in N of radius R centered at x is denoted by $B_R(x)$ and $D_R(x)=B_R(x)\cap M$. The mean value inequality [2], [7],

(1. 2) $$h(x)\leqq w_n{}^{-1}R^{-n}\int_{D_R(X)} h\,dv$$

holds for any nonnegative subharmonic function h on M. Here w_n is the n-volume of the unit n-ball in $\boldsymbol{R}^n$ and dv is the volume element of M. Since h^2 is a nonnegative subharmonic function,

$$h^2(x) \leqq w_n^{-1}R^{-n}\int_{D_R(X)} h^2 dv.$$

For a fixed point $x_0 \in M$, we denote $D_R = D_R(x_0)$. Hence

$$\operatorname{Sup}_{D_{1/2R}} g^2(x) \leqq C_1 R^{-n}\int_{D_R} g^2 dv$$

for some constant C_1 depending only on n. By varing the center of the ball and the radius, for any $\tau \in (0, \frac{1}{2})$,

$$\operatorname{Sup}_{D_{(1-\tau)R}} g^2(x) \leqq C_2 \tau^{-n} R^{-n}\int_{D_R} g^2 dv$$

where C_2 is a constant depending on n. Hence we have for any $\delta \in (0, \frac{1}{2})$, $\theta \in [\frac{1}{2}, 1-\delta]$,

$$\operatorname{Sup}_{D_{\theta R}} g^2 \leqq C_2 \delta^{-n} R^{-n}\int_{D_{(\theta+\delta)P}} g^2 dv.$$

On the other hand, we have

$$\int_{D_{(\theta+\delta)R}} g^2 dv \leqq (\operatorname{Sup}_{D_{(\theta+\delta)R}} g^{2-p})\int_{D_{(\theta+\delta)R}} g^p dv \leqq (\operatorname{Sup}_{D_{(\theta+\delta)R}} g^2)^{1-\frac{p}{2}}\int_{D_R} g^p dv,$$

since $\theta+\delta \leqq 1$.

Set
$$M(\theta) = \operatorname{Sup}_{D_{\theta R}} g^2$$
$$K = R^{-n}\int_{D_R} g^p dv.$$

Then for any $\delta \in (0, \frac{1}{2}]$, $\theta \in [\frac{1}{2}, 1-\delta]$,

$$M(\theta) \leqq C_2 K \delta^{-n} M(\theta+\delta)^{1-\frac{p}{2}}.$$

Let $\theta_0 = \frac{3}{4}$ and $\theta_i = \theta_{i-1} + 2^{-(i+2)}$ for $i = 1, 2, 3, \ldots$.

Then we have

$$M(\theta_{i-1}) \leqq K_1 2^{in} M(\theta_i)^{\lambda}$$

where $\lambda = 1-\frac{p}{2}$ and $K_1 = C_2 K 2^{2n}$.

Iterating we get

$$M(\theta_0) \leqq K_1^{\sum_{i=1}^{k} \lambda^{i-1}} 2^{n\sum_{i=1}^{k} i\lambda^{i-1}} M(\theta_k)^{\lambda^k}$$

for any $k \geqq 0$.

Letting k tend to infinity. We get

$$M(\theta_0) \leqq C_3 K^{\frac{2}{p}}$$

where C_3 is a constant depending only on n and p. Hence

$$\operatorname*{Sup}_{D_{\frac{3}{4}R}} g^p \leqq C_3 R^{-n} \int_{D_R} g^p dv.$$

Since g lies in L^p, we can let R go to infinity to show that g is identically zero.

REMARK 2. If let $h \equiv 1$ in (1.2), the volume of $D_R(x)$ is greater than or equal to the volume of ball in $\boldsymbol{R}^n$ with radius R, so M has infinite volume. Therefore for $0 < p < \infty$, every constant L^p function on M is zero.

3. The heat equation

For the heat equation, we have the following theorem.

THEOREM 3. *Let M and N be the same as in the theorem* 1. *If $v(x, t)$ is a nonnegative solution defined on $M \times [0, \infty)$ satisfying*

$$(\Delta - \frac{\partial}{\partial t}) v(x, t) \geqq 0$$

$$\int_M v(x, t) dv(x) < \infty$$

for all $t > 0$ and

$$\lim_{t \to 0} \int_M v(x, t) dv(x) = 0,$$

then $v(x, t) = 0$ for all $x \in M$ and $t \in (0, \infty)$. In particular, any L^1 solution of the heat equation on M is uniquely determined by its initial data in $L^1(M)$.

In order to prove this theorem, we have to recall the following lemma.

LEMMA 4 ([2], [7]). *Let M and N be the same as in the theorem* 1. *Let D be a compact domain in M. Given any $p \in D$, let $r_p(x)$ be the distance function on N, we denote the restriction of r_p to M as the extrinsic distance function on M and define the extrinsic outer radius at p by*

$$a = \operatorname*{Sup}_{z \in D} r_p(z).$$

Then the heat kernel $H(x, y, t)$ for the Dirichlet boundary condition on D satisfies

$$H(p, y, t) \leqq \bar{H}_a(r_p(y), t)$$

for all $y \in D$ and $t \in [0, \infty)$.

Here $\bar{H}_a(r_p(y), t)$ stands for the heat kernel with Dirichlet boundary condition on the ball centered at 0 *with radius a in* $\boldsymbol{R}^n$.

Proof of Theorem. Let $v(x,t)$ be a nonnegative function in $L^1(M)$ for all $t>0$, with

$$\lim_{t\to 0}\int_M v(x,t)\,dv(x)=0 \text{ and } \left(\Delta-\frac{\partial}{\partial t}\right)v(x,t)\geqq 0.$$

We consider the solution of the heat equation

$$e^{\Delta t}v_\varepsilon(x)=\int_M H(x,y,t)\,v(y,\varepsilon)\,dv(y)$$

with $v(x,\varepsilon)$ as initial data.
Define the function

$$G_\varepsilon(x,t)=\max\{0, v(x,t+\varepsilon)-e^{\Delta t}v_\varepsilon(x)\}.$$

Then $G_\varepsilon(x,t)$ is nonnegative and satisfies

$$\lim_{t\to 0} G_\varepsilon(x,t)=0 \text{ and } \left(\Delta-\frac{\partial}{\partial t}\right)G_\varepsilon(x,t)\geqq 0.$$

Let $g(x)=\int_0^T G_\varepsilon(x,t)\,dt$

Then $\Delta g(x)=\int_0^T \Delta G_\varepsilon(x,t)\,dt\geqq\int_0^T\frac{\partial G_\varepsilon}{\partial t}(x,t)\,dt$

$$=G_\varepsilon(x,T)-G_\varepsilon(x,0)\geqq 0$$

and

$$\begin{aligned}\int_M g(x)\,dv&=\int_M\int_0^T G_\varepsilon(x,t)\,dt\,dv(x)\\&\leqq\int_0^T\int_M |v(x,t+\varepsilon)-e^{\Delta t}v_\varepsilon(x)|\,dv(x)\,dt\\&\leqq\int_0^T\int_M|v(x,t+\varepsilon)|\,dv(x)\,dt+\int_0^T\int_M|e^{\Delta t}v_\varepsilon(x)|\,dv(x)\,dt\\&<\infty.\end{aligned}$$

The first term on the right is finite by assumption of v, and the second term is finite because $e^{\Delta t}$ is a contractive semigroup on $L^1(M)$ [8]. Thus g is a nonnegative L^1 subharmonic function on M. By Theorem 1, g is constant. Hence $G_\varepsilon(x,t)$ is identically zero, and so

(2.1) $$v(x,t+\varepsilon)\leqq e^{\Delta t}v_\varepsilon(x).$$

Consider the function

$$e^{\Delta t}v_\varepsilon(x)=\int_M H(x,y,t)\,v(y,\varepsilon)\,dv(y).$$

Then for each $x\in M$ and $t>0$,

$$e^{\Delta t}v_\varepsilon(x)\leqq\int_M(4\pi t)^{-\frac{n}{2}}\exp\left(-\frac{r_x^2(y)}{4t}\right)v(y,\varepsilon)\,dv(y)$$

$$\leqq (4\pi t)^{-\frac{n}{2}} \int_M v(y, \varepsilon)\, dv(y),$$

by Lemma 4 and the fact that for $(x, y, t) \in \boldsymbol{R}^n \times \boldsymbol{R}^n \times (0, \infty)$, $H(x, y, t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4t}}$ is a fundamental solution of the heat equation on $\boldsymbol{R}^n$. Hence $e^{At} v_\varepsilon(x) \to 0$ as $\varepsilon \to 0$. By (2.1), $v(x, t) \leqq 0$. Since v is nonnegative, $v(x, t) \equiv 0$.

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