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The Image of Derivations on Banach Algebras of Differential Functions

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ABSTRACT. Let $D : C^n(I) \longrightarrow M$ be a derivation from the Banach algebra of n times continuously differentiable functions on an interval I into a Banach $C^n(I)$-module M. If D is continuous and $D(z)$ is contained in the k-differential subspace, the image of D is contained in the k-differential subspace. The question of when the image of a derivation is contained in the k-differential subspace is discussed.

1. Introduction

Let $C^n(I), I = [0,1]$, be the algebra of all complex valued function on I which have n continuous derivatives ($n = 1, 2, \dots$). With the norm

$$\|f\|_n = \max_{t \in I} \sum_{j=0}^{n} \frac{|f^{(j)}(t)|}{j!}, f \in C^n(I),$$

$C^n(I)$ is a Banach algebra singly generated by the coordinate function z (where $z(t) = t$ for $t \in I$) and we denote the dense subalgebra of complex polynomials by β. We use the notation

$$M_{n,k}(\lambda) = \{f \in C^n(I) \mid f^{(j)}(\lambda) = 0, j = 0, 1, 2, \dots, k\}, \ \lambda \in I.$$

These are the closed ideals of finite codimension contained in the maximal ideal $M_{n,0}(\lambda)$ of functions vanishing at λ . A Banach $C^n(I)$-module is a Banach space M together with a continuous homomorphism

$$\rho : C^n(I) \longrightarrow B(M).$$

A derivation, or a module derivation of $C^n(I)$ into M is a linear map $D : \mathbf{C}^n(I) \rightarrow M$ which satisfies the identity

$$D(fg) = \rho(f)D(g) + \rho(g)D(f), \quad f,\ g \in C^n(I).$$

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Derivation from $C^n(I)$ into Banach $C^n(I)$-module have been studied by Bade, Curtis and Laursen. In [**3**], they show that the restriction of such a derivation to the subalgebra $(C^{2n}(I), ||\cdot||_{2n})$ is continuous. But discontinuous derivations from $C^n(I)$ do exist because of the existence of continuous point derivations. Note that each point derivation on $C^\infty(I)$ is continuous from Theorem 3.7. of [**4**].

Separating space $S(D)$ is the subspace of M defined by

$$S(D) = \{m \in M \mid \text{there exists}\, \{f_n\} \subset C^n(I), f_n \to 0\, \text{and}\, D(f_n) \to m\}.$$

The derivation D is continuous if and only if $S(D) = \{0\}$. The continuity ideal for a derivation $D : C^n(I) \longrightarrow M$ is

$$\Im(D) = \{f \in C^n(I) | \rho(f)S(D) = \{0\}\}.$$

Clearly $\Im(D)$ is a closed ideal in $C^n(I)$. It is proved in Theorem 3.2 of [**1**] that $\Im(D) = \{f \mid D_f \,\text{is continuous}\}$, where for $f \in C^n(I), D_f(\cdot) = \rho(f)D(\cdot)$(Note D_f is also a derivation). Let $D : C^n(I) \longrightarrow M$ be a discontinuous derivation. We can know from Theorem 4.2. of [**1**] that $\Im(D)$ has finite codimension. Let $F = \{\lambda_1, \dots, \lambda_m\}$ be the finite hull of $\Im(D)$. It follows from [**1**] that $\bigcap\{M_{n,n}(\lambda) | \lambda \in F\} \subset \Im(D)$.

THEOREM 1.1. *Let n be a positive integer, Then*

(1) $M^2_{n,0}(0) = zM_{n,0} = \{f | f(0) = f'(0) = 0$ *and* $f^{(n+1)}(0)$ *exists*$\}$,
(2) $M^2_{n,k}(0) = z^{k+1}M_{n,k}(0)$, $1 \le k \le n-1$,
(3) $M^2_{n,n}(0) = z^n M_{n,n}(0)$.

Part (1) is from example of [**1**]. Part (2) is due to Dales and McClure [**4**]. The proof of part (3) can be found in [**2**].

We shall need the following Theorem from [**2**].

THEOREM 1.2. *Let $D : C^n(I) \longrightarrow M$ be a derivation. There exists a finite set $F = \{\lambda_1, \dots, \lambda_m\}$ such that*

$$\bigcap_{i=1}^{m} M_{n,n-1}(\lambda_i) \subset \Im(D).$$

The hull F of $\Im(D)$ is called the singularity set for D. The algebras $C^n(I)$ are regular, semisimple unital commutative Banach algebras, as standard arguments readily show. Also, the primary closed ideals are easily described, by the order of which the functions in the ideal vanish. Explicity we have that since $\Phi_{C^n(I)} = I$ for every $n \geq 0$, any primary closed ideal must be of the form $M_{n,k}(\lambda)$, $k = 0, 1, \ldots, n$, $\lambda \in I$. $M_{n,k}(\lambda)$ is topologically generated as an ideal in $C^n(I)$ by $(z - \lambda)^{k+1}$.

2. $C^n(I)$-module and derivation

If M is a $C^n(I)$-module, the module structure imposes certain restrictions on the operator $\rho(z)$ and the continuity ideal of a derivation. Recall that the ascent of an eigenvalue λ for a linear operator T is the smallest integer k such that $(T - \lambda I)^{k+1}x = 0$ implies $(T - \lambda I)^k x = 0$.

DEFINITION 2.1: Let M be a Banach $C^n(I)$-module. The k-differential subspace is the set W_k $(k = 0, 1, \ldots, n)$ of all vectors m such that the map

$$p \longrightarrow \rho(p')m$$

is continuous for the $C^{n-k+1}(I)$ norm on β.

THEOREM 2.2. *$p \longrightarrow \rho(p^{(i)})m$ is continuous for the $C^n(I)$ norm on β if and only if $p \longrightarrow \rho(p^{(i+j)})m$ is continuous for the $C^{n+j}(I)$ norm on β on $(i, j = 0, 1, 2, \ldots)$.*

PROOF: For $p \in \beta$,

$$\begin{aligned} \|p^{(i)}\|_n &= \max_{t\in I} \sum_{l=0}^{n} \frac{|p^{(i+l)}(t)|}{l!} \\ &= \max_{t\in I} \sum_{l=0}^{n} \frac{|p^{(i+l)}(t)|}{(l+1)!}(l+1) \\ &\leq (n+1)\|p^{(i-1)}\|_{n+1}. \end{aligned}$$

Using the mean value theorem, for $p \in \beta \cap M_{n,j-1}$

$$\max_{t\in I} |p^{(i-1)}(t)| \leq \max_{t\in I} |p^{(i)}(t)| \quad (1 \leq i \leq j).$$

$$\begin{aligned}||p^{(i-1)}||_{n+1} &= \max_{t\in I} \sum_{l=0}^{(n+1)} \frac{|p^{(i-1+l)}(t)|}{l!} \\ &\le \max_{t\in I} |p^{(i-1)}(t)| + \max_{t\in I} \sum_{l=1}^{n+1} \frac{|p^{(i-1+l)}(t)|}{l!} \\ &\le 2||p^{(i)}||_n.\end{aligned}$$

Hence

$$(\frac{1}{2})^j ||p||_{n+j} \le ||p^{(j)}||_n \le (n+j)!||p||_{n+j}.$$

If $||\rho(p^{(i)})m|| \le L||p||_n$, $L > 0$, $p \in \beta$, then

$$||\rho(p^{(i+j)})m|| \le L||p^{(j)}||_n \le L(n+j)!||p||_{n+j}.$$

Conversely, suppose

$$||\rho(p^{(i+j)})m|| \le L||p||_{n+j},\ L > 0.$$

Let

$$q = p - p(0) - p'(0)z - \cdots - \frac{p^{(j-1)}(0)}{(j-1)!} z^{j-1}.$$

Then

$$\begin{aligned}||\rho(p^{(i+j)})m|| &= ||\rho(q^{(i+j)})m|| \\ &\le L||q||_{n+j} \\ &\le 2^j L||q^{(j)}||_n \\ &= 2^j L||p^{(j)}||_n.\end{aligned}$$

Hence

$$||\rho(p^{(i)})m|| \le 2^j L||p||_n,\ p \in \beta.$$

COROLLARY 2.3. *(1) $m \in W_k$ if and only if $p \longrightarrow \rho(p)m$ is continuous for the $C^{n-k}(I)$ norm on β. (2) $W_n \subset W_{n-1} \subset \cdots \subset W_0 = M$.*

PROOF: We have replaced n by $n-k$. Put $i = 0, j = 1$. This proves (1).

THEOREM 2.4. *Let M be a $C^n(I)$ module with k-differential subspace W_k. For $m \in W_k$, we define $|||m|||_k = \sup\{||\rho(p)m|| \mid ||p||_{n-k} \leq 1\}$.*
Then

(1) $||m|| \leq |||m|||_0 \leq |||m|||_1 \leq \cdots \leq |||m|||_k, \quad m \in W_k$.
(2) *W_k is a Banach space under the norm $|||\cdot|||_k$.*
(3) *W_k is a $C^{n-k}(I)$ module and there exists a unique continuous homomorphism*
$$\gamma_k : C^{n-k}(I) \longrightarrow B(W_k)$$
such that $\gamma_k(p)m = \rho(p)m, m \in W_k, p \in \beta$.
(4) *If $S \in B(W_i)$ and $S\rho(z) = \rho(z)S$ on W_i, $i = 0, 1, \ldots, k$, then $SW_k \subset W_k$ and $|||S|||_k \leq ||S||$, where $|||S|||_k$ is the norm of S in $B(W_k)$.*

PROOF: If $m \in W_k$,
$$||m|| = ||\rho(1)m|| \leq |||m|||_0.$$
If $||p||_{n-i} \leq 1, 0 \leq i \leq k$, $||p||_{n-j} \leq 1$, $i \leq j \leq k$. Since $W_k \subset W_{k-1} \subset \cdots \subset W_0$,
$$|||m|||_0 \leq |||m|||_1 \leq \cdots \leq |||m|||_k.$$
This proves (1). Let $\{m_j\}$ be a Cauchy sequence in W_k. By (1), $\{m_j\}$ is a Cauchy sequence in M. So $m_j \longrightarrow m_0$ in M. For $m \in W_k$, let U_m be the unique operator in $B(C^{n-k}(I), M)$ such that $U_m(p) = \rho(p)m, p \in \beta$. The operators $\{U_{m_j}\}$ are convergent in $B(C^{n-k}(I), M)$ to an operator V. Since $V(p) = \lim_{j \to \infty} U_{m_j}(p) = \rho(p)m_0$, $p \in \beta$,
$$||\rho(p)m_0|| \leq ||V|| ||p||_{n-k}.$$
Hence $m_0 \in W_k$. Since $|||m_j - m_0|||_k = ||U_{m_j} - U_{m_0}||_k \longrightarrow 0$, W_k is complete. This proves (2). Let $S \in B(W_i)$ and $S\rho(z)m = \rho(z)Sm$, for $m \in W_i$, $i = 0, 1, \ldots, m$. Then $||\rho(p)Sm|| \leq ||S|| \, |||m|||_k ||p||_{n-k}$ for $p \in \beta, m \in W_k$. So $Sm \in W_k$ and $|||S|||_k \leq ||S||$. This shows (4). If $p, q \in \beta, m \in W_k$, we have
$$||\rho(q)\rho(p)m|| = ||\rho(pq)m|| \leq |||m|||_k ||p||_{n-k} ||q||_{n-k}.$$
So $\rho(p)m \in W_k$ and $|||\rho(p)m|||_k \leq ||p||_{n-k} |||m|||_k$. Hence the map $p \longrightarrow \rho(p)m$ is a continuous homomorphism of β into $B(W_k)$ for the $C^{n-k}(I)$ norm. Its unique continuous extension $\gamma_k : C^{n-k}(I) \longrightarrow B(W_k)$ makes W_k a $C^{n-k}(I)$-module.

THEOREM 2.5. *If $\gamma_k : C^{n-k}(I) \longrightarrow B(W_k)$ is a continuous homomorphism, every eigenvalue for $\gamma_k(z)$ has ascent at most $n-k+1$.*

PROOF: Suppose $m \in W_k, (\gamma_k(z) - \lambda I)^{l+1}m = 0$, but $(\gamma_k(z) - \lambda I)^l m \neq 0$. For any polynomial p,

$$\gamma_k(p)m = \sum_{i=0}^{l} \frac{p^{(i)}(\lambda)}{i!}(\gamma_k(z) - \lambda I)^i m.$$

If $\alpha_0 m + \alpha_1(\gamma_k(z) - \lambda I)m + \cdots + \alpha_l(\gamma_k(z) - \lambda I)^l m = 0$, then $\alpha_0 = \alpha_1 = \cdots = \alpha_l = 0$. Hence $m, (\gamma_k(z) - \lambda I)m, \ldots, (\gamma_k(z) - \lambda I)^l m$ are linearly independent. Since $|||\gamma_k(p)m|||_k \leq |||m|||_k ||p||_{n-k}$, $l \leq n-k$.

THEOREM 2.6. *If D is a continuous derivation of $C^n(I)$ into M, then $D(z) \in W_1$ and $D(f) = \gamma_1(f')D(z), f \in C^n(I)$. Moreover $D(C^n(I)) \subset W_1$.*

PROOF: See Theorem 4.5 of [**2**].

THEOREM 2.7. *If $D : C^n(I) \longrightarrow M$ is a continuous derivation and $D(z) \in W_k$, then D is continuous as a derivation of $C^n(I)$ into W_k.*

PROOF: For any $f \in C^n(I)$, by Theorem 2.4,

$$D(f) = \gamma_1(f')D(z) = \gamma_k(f')D(z).$$

$$\begin{aligned} |||D(f)|||_k &= |||\gamma_k(f')D(z)|||_k \\ &\leq ||f'||_{n-k}|||D(z)|||_k \\ &\leq n||f||_n|||D(z)|||_k. \end{aligned}$$

A nontrivial derivation $D : C^n(I) \longrightarrow M$ will be singular if D vanishes on β, i.e. $D(z) = 0$.

THEOREM 2.8. *Let $D : C^n(I) \longrightarrow M$ be a discontinuous derivation with singularity set $F = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$. Then $D(z) \in W_k$ and $\bigcap_{i=1}^m M_{n,n-k}(\lambda_i) \subset \Im(D)$ iff $D(C^n(I)) \subset W_k$.*

PROOF.: Choose $e_k \in C^n(I), k = 1, \ldots, m$, such that $e_k(\lambda) = 1$ in a neighborhood of λ_k and $e_k(\lambda) = 0$ in a neighborhood of $F - \{\lambda_k\}$.

Let $e_0 = 1 - \sum_{i=1}^{m} e_i$. Then

$$e_0 \in \bigcap_{i=1}^{m} M_{n,n}(\lambda_i) \subset \Im(D),$$

$$D(f) = \sum_{i=0}^{m} \rho(e_i)D(f),\ f \in C^n(I).$$

Let $D_i(\cdot) = \rho(e_i)D(\cdot)$, D_0 is continuous and $D_i, i = 1,2,\ldots,m$, is discontinuous. We have

$$\mathrm{hull}(\Im(D_i)) = \{\lambda_i\},\ i = 1,2,\ldots,m$$

and

$$\Im(D) = \bigcap_{i=1}^{m} \Im(D_i).$$

Suppose $D(z) \in W_k$ and $\cap_{i=1}^{m} M_{n,n-k}(\lambda_i) \subset \Im(D)$. Since $e_j \in M_{n,n-k}(\lambda_i)$, $i \neq j$,

$$ge_j \in \bigcap_{i=1}^{m} M_{n,n-k}(\lambda_i),\ \text{for}\ g \in M_{n,n-k}(\lambda_j).$$

Hence

$$\begin{aligned}\rho(g)D_j(\cdot) &= \rho(g)\rho(e_j)D(\cdot)\\ &= \rho(ge_j)D(\cdot)\end{aligned}$$

is continuous. We have

$$M_{n,n-k}(\lambda_j) \subset \Im(D_j),\ j = 1,2,\ldots,m.$$

Since $D(z) \in W_k$, we have $D_j(z) \in W_k$, $j = 1,2,\ldots,m$, from Theorem 2.4. Thus it suffices to prove the theorem when the hull of $\Im(D)$ is a single point, which we may suppose to be the point zero. Since $D(z) \in W_k \subset W_1$, by Theorem 4.6 of [**2**],

$$D = E + F$$

where E is continuous and F is singular. Since $D(z) = E(z) \in W_k$, $E(C^n(I)) \subset W_k$. From $\Im(D) = \Im(F)$,

$$M_{n,n-k}(0) \subset \Im(F).$$

So $z^{n-k+1} \in \Im(F)$. For all $f \in C^n(I)$,

$$\rho(z^{n-k+1})F(f) = \gamma_1(f')\rho(z^{n-k+1})F(z) = 0.$$

For $f \in C^n(I)$, $p \in \beta$,

$$\begin{aligned} \|\rho(p)F(f)\| &= \|\rho(p(0) + p'(0)z + \cdots + \frac{p^{(n-k)}(0)}{(n-k)!}z^{n-k})F(f)\| \\ &\leq L\|p\|_{n-k},\ L > 0. \end{aligned}$$

So $F(f) \in W_k$ for all $f \in C^n(I)$.

Conversely, suppose $D(f) \in W_k$, for $f \in C^n(I)$.

$$D_i(f) = \rho(e_i)D(f) \in W_k,\ i = 0, 1, 2, \ldots, m.$$

It is sufficient to prove $z^{n-k+1} \in \Im(D)$ when the hull of $\Im(D)$ is zero. In W_k, the ascent of the eigenvalue 0 for $\rho(z)$ has at most $n-k+1$ from Theorem 2.5. From $z^n \in \Im(F)$,

$$\rho(z^n)F(f) = 0 \quad \text{implies} \quad \rho(z^{n-k+1})F(f) = 0.$$

Hence $z^{n-k+1} \in \Im(D)$, so $M_{n,n-k}(0) \subset \Im(D)$. In result,

$$M_{n,n-k}(\lambda_i) \subset \Im(D_i),\ i = 1, 2, \ldots, m.$$

Since $\Im(D) = \bigcap_{i=1}^{m} \Im(D_i)$,

$$\bigcap_{i=1}^{m} M_{n,n-k}(\lambda_i) \subset \Im(D).$$

COROLLARY 2.9. *Let $D : C^n(I) \longrightarrow M$ be a derivation. If $D(z) \in W_1$, then $D(C^n(I)) \subset W_1$.*

PROOF: If D is continuous, it is proved by Theorem 2.7. Suppose D is discontinuous. We can know $\cap_{i=1}^{m} M_{n,n-1}(\lambda_i) \subset \Im(D)$ from Theorem 1.2 where $\text{hull}(\Im(D)) = \{\lambda_1, \ldots, \lambda_m\}$.

THEOREM 2.10. *Let $D : C^n(I) \longrightarrow M$ be a derivation. Then*
(1) $D(\Im(D)^2) \subset W_1$.
(2) *If* $D(z) \in W_1$, $D(f) = \gamma_1(f')D(z)$, $f \in \Im(D)^2$.

PROOF: If $f \in \Im(D)^2$, $f = gh$ for some g, $h \in \Im(D)$.

$$D(f) = D(gh) = \rho(g)D(h) + \rho(h)D(g).$$

Since $\rho(g)D(\cdot)$ and $\rho(h)D(\cdot)$ are continuous derivation on $C^n(I)$,

$$\rho(g)D(h), \rho(h)D(g) \in W_1.$$

If $D(z) \in W_1$,

$$\begin{aligned} D(f) &= \gamma_1(h')\rho(g)D(z) + \gamma_1(g')\rho(h)D(z) \\ &= \gamma_1(h'g + g'h)D(z) \\ &= \gamma_1(f')D(z). \end{aligned}$$

COROLLARY 2.11. *Let $D : C^n(I) \longrightarrow M$ be a singular derivation, then $D(\Im(D)^2) = 0$.*

COROLLARY 2.12. *Let $D : C^n(I) \longrightarrow M$ be a derivation. If $D(z) \in W_k$, then $D(\Im(D)^2) \subset W_k$.*

PROOF: It is easily proved from Theorem 2.4.

THEOREM 2.13. *Let D be a continuous derivation from $C^n(I)$ into M. $D(z) \in W_k$ iff there exists a unique continuous derivation $D_k : C^{n-k+1}(I) \longrightarrow W_k$ such that $D_k|_{C^n(I)} = D$.*

PROOF: If $D(z) \in W_k$, by (3) of Theorem 2.4, there exists a unique continuous homomorphism

$$\gamma_k : C^{n-k}(I) \longrightarrow B(W_k)$$

such that $\gamma_k(p)m = \rho(p)m, m \in W_k, p \in \beta$. Since D is continuous,

$$D(f) = \gamma_k(f')D(z), f \in C^n(I).$$

We define $D_k : C^{n-k+1}(I) \longrightarrow M$ such that $D_k(f) = \gamma_k(f')D(z)$. Since $D(z) \in W_k$, $D_k(f) \in W_k$, by Theorem 2.4

$$\begin{aligned}|||D_k(f)|||_k &\leq ||f'||_{n-k}|||D(z)|||_k \\ &\leq (n-k+1)||f||_{n-k+1}|||D(z)|||_k.\end{aligned}$$

Conversely, suppose $D_k : C^{n-k+1}(I) \longrightarrow W_k$ is the continuous derivation such that $D_k|_{C^n(I)} = D$. Then

$$D_k(z) = D(z) \in W_k.$$

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