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C-LINDELÖF SPACES

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ABSTRACT. In this paper we introduce the notion of C-lindelöf spaces, and we discuss some of the properties that C-Lindelöf spaces satisfy.

Every closed subspace of the Lindelöf space is also Lindelöf. But a closed subspace of the W-Lindelöf space need not be W-Lindelöf. Thus we get the notion of C-Lindelöf spaces. In this paper we introduce the concept of C-Lindelöf spaces and discuss some properties of the C-Lindelöf space.

DEFINITION 1: A space X is said to be ***W-Lindelöf*** if for each open cover $\{U_i\}$ of X there exist countably many i_k such that $X = \bigcup \mathrm{Cl}(U_{i_k})$.

DEFINITION 2: A space X is said to be ***C-Lindelöf*** if for each closed subset Y of X and each open cover $\{U_i\}$ of Y there exist countably many i_k such that $Y \subset \bigcup \mathrm{Cl}(U_{i_k})$.

It is easy to show that the following theorem holds.

THEOREM 1. *Every Lindelöf space is C-Lindelöf.*

A C-Lindelöf space need not be Lindelöf as the following example shows.

EXAMPLE 1: Let $X = R \times R^+$. For $(x, y) \in X$ and $r > 0$, let

$$N_r(x,y) = \begin{cases} B_r(x,y) & \text{if } r \leq y \\ B_r(x,r) \cup \{(x,0)\} \cup B_r(0,r) & \text{if } y = 0 \end{cases}$$

We take $\{N_r(x,y)\}$ as a basis for the topology on X.

We shall show that X is C-Lindelöf. For a closed subset Y of X, let $Y = Y_1 \cup Y_2$ where $Y_1 = Y \cap (R \times \{0\})$ and $Y_2 = Y - Y_1$. We

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may assume that $Y_1 \neq \emptyset$. Let $(a,0) \in Y_1$. Given an open cover $\mathcal{U}$ of Y, there exists $U \in \mathcal{U}$ such that $(a,0) \in U$. We can choose $r > 0$ so that $N_r(a,0) \subset U$. We claim that $Y_1 \subset \mathrm{Cl}(U)$. Let $(x,0) \in Y_1$. For any $0 < s \le r$, $B_s(0,s) \subset N_s(x,0) \cap N_r(a,0) \subset N_s(x,0) \cap U$. Thus $(x,0) \in \mathrm{Cl}(U)$. Let $\mathcal{A} = \{A \in \mathcal{B} :$ there exists $V \in \mathcal{U}$ such that $A \subset V\}$ where $\mathcal{B} = \{B_r(x,y) : x,\ y,\ r \in Q$ and $r \le y\}$. It is clear that $\mathcal{A}$ is countable. Let $\mathcal{A} = \{A_n\}$. For each n, there exists $V_n \in \mathcal{U}$ such that $A_n \subset V_n$. We claim that $Y_2 \subset \bigcup V_n$. Let $(x,y) \in Y_2$. There exists $V \in \mathcal{U}$ such that $(x,y) \in V$. We can choose $r \in Q$ so that $0 < r \le y/2$ and $B_{2r}(x,y) \subset V$. There exist $b,\ c \in Q$ such that $(b,c) \in B_r(x,y)$. Since $B_r(b,c) \subset B_{2r}(x,y) \subset V$, $B_r(b,c) \in \mathcal{A}$. Thus there exists n_1 such that $B_r(b,c) = A_{n_1}$. Then $(x,y) \in B_r(b,c) = A_{n_1} \subset V_{n_1}$. Hence $Y \subset \bigcup V_n \cup \mathrm{Cl}(U)$ and so X is C-Lindelöf.

Let us show that X is not Lindelöf. $\{N_1(x,0)\} \cup \{N_1(x,y) : y \ge 1\}$ is an open cover of X. Since $(z,0) \notin N_1(x,y)$ if $y \ge 1$ and $(z,0) \in N_1(x,0)$ if and only if $x = z$, above open cover has no countable subcover. Thus X is not Lindelöf.

Every C-Lindelöf space is W-Lindelöf, but its converse does not hold as the following example shows.

EXAMPLE 2: Let $X = R \times R^+$. For $(x,y) \in X$ and $r > 0$, let

$$N_r(x,y) = \begin{cases} B_r(x,y) & \text{if } r \le y \\ \bigcup_{z\in Q} B_r(z,r) \cup \{(x,0)\} & \text{if } x \in Q \text{ and } y = 0 \\ B_r(x,r) \cup \{(x,0)\} & \text{if } x \in R - Q \text{ and } y = 0 \end{cases}$$

We take $\{N_r(x,y)\}$ as a basis for the topology on X.

We shall show that X is W-Lindelöf. Let $\mathcal{U}$ be an open cover of X. There is $U \in \mathcal{U}$ such that $(0,0) \in U$. We claim that $R \times \{0\} \subset \mathrm{Cl}(U)$. There exists $a > 0$ such that $N_a(0,0) \subset U$. Let $x \in R$. For each $0 < b \le a$, we can choose $y \in Q$ so that $|x - y| < 2\sqrt{ab}$. Since $\sqrt{(x-y)^2 + (b-a)^2} < a + b$,

$$\emptyset \neq B_b(x,b) \cap B_a(y,a) \subset N_b(x,0) \cap N_a(0,0) \subset N_b(x,0) \cap U.$$

Thus $(x,0) \in \mathrm{Cl}(U)$. Let $\mathcal{A} = \{A \in \mathcal{B} :$ there exists $V \in \mathcal{U}$ such that $A \subset V\}$ where $\mathcal{B} = \{B_r(x,y) : x,\ y,\ r \in Q$ and $r \le y\}$. Then $\mathcal{A}$ is countable. Let $\mathcal{A} = \{A_n\}$. For each n, there exists $V_n \in \mathcal{U}$ such that

$A_n \subset V_n$. We claim that $R \times (0,\infty) \subset \bigcup V_n$. Let $(x,y) \in R \times (0,\infty)$. There exists $V \in \mathcal{U}$ such that $(x,y) \in V$. We can choose $r \in Q$ so that $0 < r \leq y/2$ and $B_{2r}(x,y) \subset V$. There exist a, $b \in Q$ such that $(a,b) \in B_r(x,y)$. Since $B_r(a,b) \subset B_{2r}(x,y) \subset V$, $B_r(a,b) \in \mathcal{A}$. Thus there exists n_1 such that $B_r(a,b) = A_{n_1}$. Then $(x,y) \in B_r(a,b) = A_{n_1} \subset V_{n_1}$. Thus $X = \bigcup V_n \cup \mathrm{Cl}(U)$ and so X is W-Lindelöf.

We shall show that X is not C-Lindelöf. Let $Y = (R-0) \times \{Q\}$. Then Y is a closed subset of X. $\{N_1(x,0) : x \in R - Q\}$ is an open cover of Y. We claim that $\mathrm{Cl}(N_1(a,0)) = B_1[a,1] \cup (Q \times \{0\})$ for all $a \in R - Q$. Let $(x,y) \in X$.

(1) $(x-a)^2 + (y-1)^2 > 1$ and $y > 0$. Let

$$r = \min(y, \sqrt{(x-a)^2 + (y-1)^2} - 1).$$

Since $\sqrt{(x-a)^2 + (y-1)^2} \geq r + 1$, $B_r(x,y) \cap N_1(a,0) = \emptyset$. Thus $(x,y) \in X - \mathrm{Cl}(N_1(a,0))$.

(2) $(x-a)^2 + (y-1)^2 = 1$ and $y > 0$. For any $0 < r \leq y$, since $\sqrt{(x-a)^2 + (y-1)^2} < r + 1$, $B_r(x,y) \cap N_1(a,0) \neq \emptyset$. Thus $(x,y) \in \mathrm{Cl}(N_1(a,0))$.

(3) $x \in R - (Q \cup \{a\})$ and $y = 0$. Let $0 < r < (x-a)^2/4$. Since $\sqrt{(x-a)^2 + (r-1)^2} > r + 1$, $N_r(x,0) \cap N_1(a,0) = \emptyset$. Thus $(x,0) \in X - \mathrm{Cl}(N_1(a,0))$.

(4) $x \in Q$ and $y = 0$. For any $r > 0$, there exists $b \in Q$ such that $|a - b| < 2\sqrt{r}$.

Since $\sqrt{(a-b)^2 + (1-r)^2} < r+1$, $\emptyset \neq B_r(b,r) \cap B_1(a,1) \subset N_r(x,0) \cap N_1(a,0)$. Thus $(x,0) \in \mathrm{Cl}(N_1(a,0))$. Since $(x,0) \in \mathrm{Cl}(N_1(a,0))$ if and only if $x = a$, $Y \not\subset \bigcup \mathrm{Cl}(N_1(x_i,0))$ for all countably many x_i. Hence X is not C-Lindelöf.

THEOREM 2. *Let X be a space. If every closed subspace of X is W-Lindelöf then X is C-Lindelöf.*

PROOF: Let Y be a closed subset of X and let $\{U_i\}$ be an open cover of Y. Since $\{U_i \cap Y\}$ is a cover of Y by sets open in Y, there exist countably many i_k such that $Y = \bigcup \mathrm{Cl}_Y(U_{i_k} \cap Y)$. Since $\mathrm{Cl}_Y(U_{i_k} \cap Y) = \mathrm{Cl}(U_{i_k} \cap Y) \cap Y \subset \mathrm{Cl}(U_{i_k})$ for all i_k, $Y \subset \bigcup \mathrm{Cl}(U_{i_k})$. Thus X is C-Lindelöf.

The converse of above theorem does not hold as the following example shows.

EXAMPLE 3: In example 1 let $Y = R \times \{0\}$. Then Y is a closed subspace of X. Since $N_1(x,0) \cap Y = \{(x,0)\}$ for all $x \in R$, Y is an uncountable discrete space. Thus Y is not W-Lindelöf.

THEOREM 3. *Let X be a regular space. Then the following statements are equivalent.*

(1) *X is Lindelöf.*
(2) *X is C-Lindelöf.*
(3) *X is W-Lindelöf.*

PROOF: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) It is clear.

(3) $\Rightarrow$ (1) Let $\mathcal{U}$ be an open cover of X. For each $x \in X$ there exists $U_x \in \mathcal{U}$ such that $x \in U_x$. Since X is regular, there exists a neighborhood V_x of x such that $\mathrm{Cl}(V_x) \subset U_x$. $\{V_x : x \in X\}$ is an open cover of X. Since X is W-Lindelöf, there exist countably many $x_i \in X$ such that

$$X = \cup \mathrm{Cl}(V_{x_i}) \subset \cup U_{x_i}.$$

Thus X is Lindelöf.

THEOREM 4. *Let X be a locally compact Hausdorff space. Then X is C-Lindelöf if and only if X is σ-compact.*

PROOF: ($\Rightarrow$) For each $x \in X$ there exists a neighborhood U_x of x such that $\mathrm{Cl}(U_x)$ is compact. $\{U_x : x \in X\}$ is an open cover of X. Since X is C-Lindelöf, there exist countably many $x_i \in X$ such that $X = \bigcup \mathrm{Cl}(U_{x_i})$. Thus X is σ-compact.

($\Leftarrow$) Let $X = \bigcup X_n$ and X_n compact. Given a closed subset Y of X, let $Y_n = Y \cap X_n$. Then $Y = \bigcup Y_n$ and Y_n is compact. For any open cover $\mathcal{U}$ of Y, since $\mathcal{U}$ is an open cover of Y_n, there eixst finitely many $U_{n,i} \in \mathcal{U}$ such that $Y_n \subset \bigcup U_{n,i}$. Then $Y = \bigcup Y_n \subset \bigcup \bigcup U_{n,i}$. Thus X is C-Lindelöf.

THEOREM 5. *A space X is C-Lindelöf if and only if for each closed subset Y of X and each regular open cover $\{U_i\}$ of Y there exist countably many i_k such that $Y \subset \bigcup \mathrm{Cl}(U_{i_k})$.*

PROOF: ($\Rightarrow$) It is clear.

($\Leftarrow$) Let Y be a closed subset of X. Given any open cover $\{U_i\}$ of Y, since $\{\mathrm{Int}(\mathrm{Cl}(U_i))\}$ is a regular open cover of Y, there exist countably many i_k such that $Y \subset \bigcup \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(U_{i_k}))) = \bigcup \mathrm{Cl}(U_{i_k})$. Thus X is C-Lindelöf.

THEOREM 6. *A space X is C-Lindelöf if and only if for each closed subset Y of X and each family $\{A_i\}$ of closed subsets of X such that $\bigcap \mathrm{Int}(A_{i_k}) \cap Y \neq \emptyset$ for all countably many i_k, $\bigcap A_i \cap Y \neq \emptyset$.*

PROOF: ($\Rightarrow$) Suppose that $\bigcap A_i \cap Y = \emptyset$. Since $Y \subset X - \bigcap A_i = \bigcup(x - A_i)$, $\{X - A_i\}$ is an open cover of Y. Since X is C-Lindelöf, there exist countably many i_k such that $Y \subset \bigcup \mathrm{Cl}(X - A_{i_k}) = X - \bigcap(X - \mathrm{Cl}(X - A_{i_k})) = X - \bigcap \mathrm{Int}(A_{i_k})$. Then $\bigcap \mathrm{Int}(A_{i_k}) \cap Y = \emptyset$, this is a contradiction. Hence $\bigcap A_i \cap Y \neq \emptyset$.

($\Leftarrow$) Let Y be a closed subset of X and let $\{U_i\}$ be an open cover of Y. Suppose that $Y \not\subset \bigcup \mathrm{Cl}(U_{i_k})$ for all countably many i_k. $\{X - U_i\}$ is a family of closed subsets of X and

$$\begin{aligned}\cap \mathrm{Int}(X - U_{i_k}) \cap Y &= \cap(X - \mathrm{Cl}(U_{i_k})) \cap Y \\ &= (X - \cup \mathrm{Cl}(U_{i_k})) \cap Y \neq \emptyset\end{aligned}$$

for all countably many i_k. Thus $(X - \bigcup U_i) \cap Y = \bigcap(X - U_i) \cap Y \neq \emptyset$. Then $Y \not\subset \bigcup U_i$, this is a contradiction. Hence there exist countably many i_k such that $Y \subset \bigcap \mathrm{Cl}(U_{i_k})$ and so X is C-Lindelöf.

DEFINITION 3: Let X be a space. A filter $\mathcal{A}$ in $Y \subset X$ is said to ***r-accumulate*** to $x \in Y$, denoted by $\mathcal{A} \overset{r}{\propto} x$, if for each neighborhood U of x and each $A \in \mathcal{A}$, $A \cap \mathrm{Cl}(U) \neq \emptyset$.

DEFINITION 4: Let X be a set. A faimly $\mathcal{A}$ of nonempty subsets of X is said to be a ***δ-filter*** in X if

(1) if $A_1, A_2, \cdots \in \mathcal{A}$, then $\bigcap A_n \in \mathcal{A}$,
(2) if $A \subset B \subset X$ for some $A \in \mathcal{A}$, then $B \in \mathcal{A}$.

THEOREM 7. *A space X is C-Lindelöf if and only if for each closed subset Y of X and each δ-filter $\mathcal{A}$ in Y there exists $x \in Y$ such that $\mathcal{A} \overset{r}{\propto} x$.*

PROOF: ($\Rightarrow$) Suppose that $\mathcal{A} \overset{r}{\not\propto} x$ for all $x \in Y$. For each $x \in Y$ there exist a neighborhood U_x of x and $A_x \in \mathcal{A}$ such that $A_x \cap \mathrm{Cl}(U_x) = \emptyset$. Then $A_x \subset X - \mathrm{Cl}(U_x)$ for all $x \in Y$ and $\{U_x\}$ is an open cover of Y. Since X is C-Lindelöf, there exist countably many $x_i \in Y$ such that $Y \subset \bigcup \mathrm{Cl}(U_{x_i})$. Since

$$\cap A_{x_i} \subset \cap(X - \mathrm{Cl}(U_{x_i})) = X - \cup \mathrm{Cl}(U_{x_i}) \subset X - Y \text{ and } \cap A_{x_i} \subset Y,$$

$\bigcap A_{x_i} = \emptyset$. This is a contradiction for $\bigcap A_{x_i} \in \mathcal{A}$. Thus there exists $x \in Y$ such that $\mathcal{A} \overset{r}{\propto} x$.

($\Leftarrow$) Let Y be a closed subset of X and let $\{U_i\}$ be an open cover of Y. Suppose that $Y \not\subset \bigcup \mathrm{Cl}(U_{i_k})$ for all countably many i_k. Let $\mathcal{A}$ be the family of all subsets of Y containing $\bigcap(X - \mathrm{Cl}(U_{i_k})) \cap Y$ for some countably many i_k. Then $\mathcal{A}$ is a δ-filter in Y and $(X - \mathrm{Cl}(U_i)) \cap Y \in \mathcal{A}$ for all i. There exists $x \in Y$ such that $\mathcal{A} \overset{r}{\propto} x$. There exists i_1 such that $x \in U_{i_1}$. Since $(X - \mathrm{Cl}(U_{i_1})) \cap Y \in \mathcal{A}$, $\mathrm{Cl}(U_{i_1}) \cap (X - \mathrm{Cl}(U_{i_1})) \cap Y \neq \emptyset$. This is a contradiction. Thus there exist countably many i_k such that $Y \subset \bigcup \mathrm{Cl}(U_{i_k})$ and so X is C-Lindelöf.

THEOREM 8. *If X is C-Lindelöf and $f : X \to Y$ is a continuous surjection, then Y is C-Lindelöf.*

PROOF: Let A be a closed subset of Y and let $\{U_i\}$ be an open cover of A. $f^{-1}(A)$ is a closed subset of X and $\{f^{-1}(U_i)\}$ is an open cover of $f^{-1}(A)$. Since X is C-Lindelöf, there exist countably many i_k such that $f^{-1}(A) \subset \bigcup \mathrm{Cl}(f^{-1}(U_{i_k}))$. Since

$$\begin{aligned} A &= ff^{-1}(A) \subset f(\cup \mathrm{Cl}(f^{-1}(U_{i_k}))) \\ &= \cup f(\mathrm{Cl}(f^{-1}(U_{i_k}))) \subset \cup \mathrm{Cl}(ff^{-1}(U_{i_k})) = \cup \mathrm{Cl}(U_{i_k}), \end{aligned}$$

Y is C-Lindelöf.

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