JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 2, June 1989

A Characterization of the Weak*-Integral

RHIE, GIL-SEOB AND PARK, HI-KYO

ABSTRACT. The main goal of the present paper is to characterize the weak*-integral, which is a weak* analogy of Geitz[4]

1. Introduction

Let (Ω, Σ, μ) be a finite measure space and X a Banach space with continuous dual X^*. An X^*-valued function f defined on Ω is said to be weak*-measurable if $\hat{x} \circ f$ is measurable for each x in X. A weak*-measurable function f is said to be weak*-integrable if $\hat{x} \circ f$ is integrable for each x in X and the weak*-integral of f over E in Σ means the element x_E^* of X^* such that

$$x_E^*(x) = \int_E \hat{x} \circ f \, d\mu$$

for all x in X. We write $x_E^* = (w^*) - \int_E f \, d\mu$.

The main goal of the present paper is to give a characterization of the weak*-integral, which is a weak* analogy of Geitz[4].

Most of the notations and terminologies follow those of Diestel and Uhl[2].

2. Main Result

For a subset A of X^*, the weak* closure of A in X^* and the weak* closed convex hull of A in X^* are denoted by $w^* - \text{cl}(A)$ and $w^* - \text{clco}(A)$, respectively. the following lemma provides the basis for this section. We will call it the mean value theorem for the weak*-integral.

Received by the editors on 17 June 1989.
1980 *Mathematics subject classifications*: Primary 28B05.

LEMMA 1. *The mean value theorem for the weak*-integral. Let $f : \Omega \to X^*$ be weak*-integrable. Then for each measurable set E of positive measure*

$$\frac{1}{\mu(E)}\left((w^*) - \int_E f\,d\mu\right) \in w^* - \text{clco}(f(E)).$$

PROOF: Suppose there is a measurable set E of positive measure such that

$$\frac{1}{\mu(E)}\left((w^*) - \int_E f\,d\mu\right) \notin w^* - \text{clco}(f(E)).$$

Then by the Hahn-Banach theorem and the fact that $(X^*, \text{weak}^*)^* = X$ [**1**, Theorem V.1.3], there exist an x in X, a real number α such that

$$\frac{1}{\mu(E)}\int_E \hat{x}\circ f\,d\mu < \alpha < \hat{x}\circ f(w)$$

for all w in E.

Integrating over E,

$$\int_E \hat{x}\circ f\,d\mu < \alpha\mu(E) < \int_E \hat{x}\circ f\,d\mu$$

a contradiction. This completes the proof.

As a weak* version of Geitz[4], we characterize the weak*-integral with the mean value theorem for the weak*-integral.

DEFINITION 2: Let $f : \Omega \to X^*$ be a weak* measurable function, B a measurable subset of Ω and π a finite partition of B into measurable sets. We define the $S^*(\pi, B)$ to be the weak*-closure of the convex set

$$\sum_{E\in\pi} w^* - \text{clco}(f(E))\mu(E).$$

REMARK: Suppose that $f : \Omega \to X^*$ is weak*-integrable. If B is measurable and π is a finite partition of B into measurable sets, then

$$(w^*) - \int_B f\,d\mu = \sum_{E\in\pi}(w^*) - \int_E f\,d\mu \in \sum_{E\in\pi} w^* - \text{clco}(f(E))\mu(E).$$

In particular, $(w^*) - \int_B f\,d\mu \in \bigcap_\pi S^*(\pi, B)$ where the intersection is taken over all partition π of B.

THEOREM 3. *Let $f : \Omega \to X^*$ be a bounded weak*-measurable function. Then $\bigcap_\pi S^*(\pi, B) = \{(w^*) - \int_B f\, d\mu\}$ for each measurable set B.*

PROOF: The above remark shows that $(w^*) - \int_B f\, d\mu \in \bigcap_\pi S^*(\pi, B)$. We will show that $(w^*) - \int_B f\, d\mu$ is the only possible element of $\bigcap_\pi S^*(\pi, B)$. Fix a measurable set B and let $x^* \in \bigcap_\pi S^*(\pi, B)$; we shall show that

$$x^*(x) = \int_B \hat{x} \circ f\, d\mu$$

for every x in X.

Let $\varepsilon > 0$ and $x \in X$. Since $\hat{x} \circ f$ is bounded, it is the uniform limit of a sequence of simple functions. It follows that there exists a finite partition π of B such that if E is in π and w_1, w_2 in E, then

$$|\hat{x} \circ f(w_1) - \hat{x} \circ f(w_2)| < \varepsilon.$$

Hence for any element w in E, we have

$$\left| \frac{1}{\mu(E)} \int_E \hat{x} \circ f\, d\mu - \hat{x} \circ f(w) \right| < \varepsilon$$

and $\left| \int_E \hat{x} \circ f\, d\mu - \hat{x} \circ f(w)\mu(E) \right| < \varepsilon \mu(E)$. If $\sum t_i f(w_i)$ is a convex sum with each w_i in E, then

$$\left| \int_\varepsilon \hat{x} \circ f\, d\mu - \sum t_i \hat{x} \circ f(w_i)\mu(E) \right| < \varepsilon \mu(E).$$

It follows that if $x^*_E \in w^* - \text{clco}(f(E))$ for each E in π, then

$$\left| \int_E \hat{x} \circ f\, d\mu - x^*_E(x)\mu(E) \right| \le \varepsilon \mu(E).$$

Summing over all the set E in π gives

$$\left| \int_B \hat{x} \circ f\, d\mu - x^*(x) \right| \le \varepsilon \mu(B).$$

Since this holds for every $\varepsilon > 0$, we have

$$x^*(x) = \int_B \hat{x} \circ f\, d\mu$$

and $x^* = (w^*) - \int_B f\, d\mu$. This completes the proof.

LEMMA 4. *Let f be a bounded weak*-measurable function. Then the vector measure $F : \Sigma \to X^*$ defined by $F(E) = (w^*) - \int_E f\, d\mu$ is countably additive.*

PROOF: Since f is bounded, it is clear that the set $S = \{\hat{x} \circ f : \|x\| \leq 1,\, x \in X\} \subset L^1(\mu)\}$ is uniformly integrable. Let E be the disjoint union of E_n's in Σ. Since S is uniformly integrable, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\int_A |\hat{x} \circ f|\, d\mu < \varepsilon$ for each x in X with $\|x\| \leq 1$ if $\mu(A) < \delta$. It follows that $\|F(A)\| < \varepsilon$. Hence $\Sigma F(E_i)$ converges to $F(E)$ in norm topology, that is F satisfies the countable additivity. This completes the proof.

COROLLARY 5. *Let $f : \Omega \to X^*$ be a weak*-measurable function. If f is bounded, then for each measurable set B, there exists a weakly compact subset W of X^* such that $W \cap S^*(\pi, B)$ is not empty for every finite partition π of B.*

PROOF: From the boundedness of f, the vector measure F defined in the above lemma is countable additive. Let W be the closure of the set $\{(w^*) - \int_E f\, d\mu : E \in \Sigma\}$ with respect to the $\sigma(X^*, X^{**})$-topology. This set is well known to be weakly compact [**2**, Corollary I.2.7] and we have already seen that $W \cap S^*(\pi, B)$ is not empty for every finite partition π of B, this completes the proof.

REFERENCES

1. J.B. Conway, "A Course in Functional Analysis," Springer Verlag, New York and Berlin, 1985.
2. J. Diestel and J.J. Uhl, Jr., *Vector measures*, in "Math. Surverys," No. 15, Amer. Math. Soc., Providence, 1977.
3. N. Dunford and J.T. Schwarz, "Linear Operator," Part I, Interscience, New York, 1958.
4. R.F. Geitz, *Geometry and the Pettis Integral*, Trans. Amer. Math. Soc. **269** (1982), 535–548.
5. M. Talagrand, *Pettis Integral and Measure Theory*, in "Memoirs No. 307," Amer. Math. Soc., Providence, 1985.

Department of Mathematics
Hannam University
Taejon, 300-791, Korea
and
Joon Kyoung Technical Junior college
Taejon, 300-100, Korea